MATHEMATICAL TRIPOS

1918—1922

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MATHEMATICAL TRIPOS

PART I

1918

THURSDAY, 30 *May*. 9—12.

1. Shew that the locus of a point, such that the lengths of the tangents from it to two circles are equal, is a straight line.

AB, CD are diameters of two circles and AC is parallel to BD. Prove that AD, BC meet on the radical axis of the circles.

2. Prove that the orthogonal projection of a circle is an ellipse.

Shew that orthogonal projection does not alter the relative areas of different parts of a figure.

Prove that in any of the triangles of maximum area inscribed in an ellipse, the tangent at each angular point is parallel to the opposite side, and that the medians intersect in the centre of the ellipse.

3. Shew that, if

$x^2 = y^2 + z^2 + 2ayz, \quad y^2 = z^2 + x^2 + 2bzx, \quad z^2 = x^2 + y^2 + 2cxy,$

then

$$\frac{x^2}{1-a^2} = \frac{y^2}{1-b^2} = \frac{z^2}{1-c^2}.$$

Prove that if x, y, z are not all equal, and

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = \lambda,$$

then $x^2y^2z^2 = 1$ and $\lambda^2 = 1$.

4. Shew that

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \quad\right\}.$$

Hence prove that $\log_{10} 13 = 1 + 3\log_{10} 11 - 10\log_{10} 2$ approximately, and that the error is about $\cdot 0000653$. [$\log_{10} e = \cdot 4343$.]

5. Prove geometrically the formula for $\tan(A+B)$, assuming that $A+B$ is less than $\frac{1}{2}\pi$.

If $A+B+C=\pi$, prove that

$$\Sigma \tan A \cot B \cot C = \Sigma \tan A - 2\Sigma \cot A.$$

Also if $\tan y = \dfrac{n \sin x \cos x}{1 - n \sin^2 x}$, prove that

$$\tan(x-y) = (1-n)\tan x.$$

6. A straight racing track AB is viewed from a point P distant 1000 feet from the half-way post H. The two halves of the track AH, HB subtend angles of 22° and 28° respectively at P. Determine within 3 inches the length of the track, and find the angles PAB, PBA.

7. Prove that, if y is an implicit function of x satisfying the equation $\phi(x, y) = 0$, the derivative of y with respect to x is given by

$$\frac{dy}{dx} = -\frac{\partial\phi}{\partial x}\bigg/\frac{\partial\phi}{\partial y}.$$

If x, y, z are connected by two equations of the form $f(x, y, z) = 0$, $\phi(x, y, z) = 0$, determine the expression for $\dfrac{dy}{dx}$.

Hence find the value of $\dfrac{dB}{dA}$ for a triangle whose angles A, B, C are such that $\Sigma \sin B \sin C = h$.

8. Integrate $\displaystyle\int \frac{(x^2+4)\,dx}{x^2+2x+3}$, $\displaystyle\int \frac{dx}{x^2(1+x^2)^2}$.

Express $ax^2+2bx+c$ in the form

$$\lambda(Ax^2+2Bx+C) + 2(\mu x+\nu)(Ax+B),$$

where A, B, C are given. Hence shew that, if $\lambda+\mu=0$,

$$\int \frac{(ax^2+2bx+c)\,dx}{(Ax^2+2Bx+C)^2} = \frac{ax+b}{A(Ax^2+2Bx+C)},$$

and that the condition $\lambda+\mu=0$ is equivalent to $Ac+Ca=2Bb$.

9. Prove that the area enclosed by the curves $xy^2=a^2(a-x)$, $(a-x)y^2=a^2x$ is $(\pi-2)a^2$.

Prove also that the volume obtained by rotating the above area round the line $x=\frac{1}{2}a$ is $\frac{1}{4}\pi a^3(4-\pi)$.

10. Find an expression for the length of the perpendicular from the point (x, y, z) upon the line $\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n}$.

Perpendiculars are drawn from one corner of a rectangular block upon the three diagonals which do not pass through that corner. Prove that if these perpendiculars are projected on any edge of the block, one of the projections is equal to the sum of the other two.

THURSDAY, 30 *May*. 2 - 5.

1. Two tangents PT, PT' are drawn touching a circle in T, T' and intersecting in P. Tangents at T_1, T_2, T_3,... cut PT, PT' in ranges of points A_1, A_2, A_3, .. and A_1', A_2', A_3',... respectively. Shew that these ranges are projective with one another, and state what point in the first range corresponds to the point P in the second.

Shew also that the two ranges are projective with the pencil formed by joining T_1, T_2, T_3, ... to any point on the circle.

2. Shew how to draw a straight line which shall intersect each of two given non-intersecting straight lines at right angles.

A straight line meets two non-intersecting perpendicular straight lines in A, B. Shew that if AB is of constant length, the locus of its middle point is a circle.

3. Write down the expansion of $(1+x)^n$ in ascending powers of x, and shew that, approximately, when x, y, z are small

$$\frac{(1+x)^m (1+y)^n}{(1+z)^p} = 1 + mx + ny - pz.$$

The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $s^3 D^{\frac{2}{3}} t^{-2}$; find approximately the percentage increase of work necessary when the distance is increased $1^\circ/_\circ$, the time is diminished $1^\circ/_\circ$ and the displacement of the ship is diminished $3^\circ/_\circ$.

Use logarithm tables to verify your result and to determine more accurately the percentage increase of work.

4. Prove that the arithmetic mean of any number of positive quantities is not less than their geometric mean.

If $a_1, a_2, \ldots a_n$ are positive numbers such that

$$a_1 + a_2 + \ldots + a_n \leqslant 1,$$

prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \geqslant n^2.$$

Prove also that if x is greater than any of the numbers $a_1, a_2, \ldots a_n$, then

$$\frac{1}{x - a_1} + \frac{1}{x - a_2} + \ldots + \frac{1}{x - a_n} \geqslant \frac{n}{x - \dfrac{a_1 + a_2 + \ldots + a_n}{n}}.$$

5. Find correct to 1°/ₒ the length of a chord which divides the area of a circle of 2 feet diameter in the ratio 1 : 2.

6. A bar OA of length a revolves uniformly about its end O, and has attached to the end A a bar AB of length b, whose end B moves to and fro in a straight line. Shew that if B completes its movement on one side of the middle point of its travel in one-half the time that OA takes to make a complete revolution, the line of travel of B is distant $(\frac{1}{2}b^2 - a^2)^{\frac{1}{2}}$ from O, and the length of the complete travel of B is $4\sqrt{2}a\,(b - \sqrt{2}a)$.

7. Prove that the line $(x - a)\cos\theta + y\sin\theta = b$ always touches the circle $(x - a)^2 + y^2 = b^2$, and find the point of contact.

A pair of parallel tangents is drawn to one of two equal circles, and another pair of tangents perpendicular to the first pair is drawn to the other circle. Prove that each of the diagonals of the square formed by the four tangents passes through a fixed point.

8. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

From a point $P\ (at^2, 2at)$ on the parabola $y^2 = 4ax$ two chords PQ, PR are drawn, normal to the curve at Q and R. Prove that the equation of QR is

$$yt + 2(x + 2a) = 0.$$

9. Prove that the locus of the centres of parallel sections of an ellipsoid is a straight line, and give the equation to this line in terms of the semi-axes a, b, c and the direction cosines l, m, n of the normal to the section planes.

Determine the locus of the centres of all sections of the ellipsoid by planes passing through a given point.

10. Find the differential equation satisfied by x, y independently of the special values of a, b in the equation

$$y = ax\cos\left(\frac{n}{x}+b\right).$$

Solve the equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-xy=0.$$

FRIDAY, 31 *May*. 9—12.

1. A number of parallel coplanar forces act at fixed points. Shew that their resultant also passes through a fixed point however the forces are turned round their points of application, provided they always remain parallel and unaltered in magnitude.

Within a triangle ABC a point P is chosen, and the line AP is produced to cut BC in D. Equal parallel forces in the plane of the triangle act at A, B, C, P, and an equal but opposite parallel force acts at D. Find the position of P in order that their resultant may always pass through P, independently of the direction of the forces.

2. A chain of four equal heavy rods, each of weight W, is held up at points A, B in the same horizontal line. If the points of suspension are drawn apart until the horizontal components of the pulls at A, B are each $2W$, shew that $AB = 3{\cdot}54$ times the length of a rod, and determine the slope of each rod.

3. Explain and prove the principle of virtual work as applied to a system of rigid bodies.

A weight W is supported by a tripod standing on the ground. Each leg of the tripod is of length l and weight w, and the feet form an equilateral triangle of side a. Equilibrium is maintained by three inextensible strings of equal length joining the middle points of the legs. Shew that when the ground is smooth the tension in the strings is

$$\frac{2W+3w}{3\sqrt{3}}\frac{a}{\sqrt{(3l^2-a^2)}},$$

but that if the ground is rough, this tension may be reduced by $2\mu\dfrac{W+3w}{3\sqrt{3}}$, where μ is the coefficient of friction.

4. Power is transmitted from one pulley to another by means of a belt round both pulleys. Shew that if the tensions in the tight and slack sides of the belt are T_1, T_2 pounds weight, and the speed of the belt is v feet per second, the horse-power transmitted is $(T_1 - T_2)v/550$.

Power is delivered by an engine through ropes passing round the flywheel of the engine and over pulleys in the mill. If the flywheel is 30 feet in diameter and turns at 90 revolutions per minute, determine how many ropes of $1\frac{1}{2}$ inch diameter will be required to transmit 2000 horse-power. Assume the tension on the tight side to be equal to twice the tension on the slack side, and allow not more than 300 pounds per square inch pull in the ropes.

5. A body starting with finite acceleration moves so as to have the following velocities in miles per hour at successive points 400 yards apart:

0, 9, 14, 16, 16, 15, 13·5, 12.

Construct acceleration-distance and time-distance curves for the motion of the body, noting carefully the units and scales used.

6. Prove that, if a particle is describing an orbit under the influence of a central force, the force F at any point of the orbit is related to the radius r and the perpendicular p on the tangent by the equation

$$F = -\tfrac{1}{2}h^2 \frac{d}{dr}\left(\frac{1}{p^2}\right),$$

where $h = pv$, and v is the velocity.

A particle is projected from a point P in a field of attractive force μ/r^5, where r is the distance from the fixed centre of attraction O. Prove that the orbit will be a circle passing through O, provided that the velocity of projection is $(\frac{1}{2}\mu)^{\frac{1}{2}}/OP^2$, and find the centre of the circle.

7. The readings of a perfect and a faulty barometer are compared on two occasions, and it is found that the differences are x_1 and x_2 inches, the heights in the perfect barometer being h_1 and h_2 respectively. The differences being assumed due to the presence of air in the second barometer, find the correction to a reading h on the latter. No correction for the limited capacity of the reservoir need be made.

8. State the laws of refraction of light, and find equations to determine the deviation of a ray which passes through a prism in a principal plane.

The angle of a glass prism, of refractive index $\frac{3}{2}$, is 50°. Shew that the greatest deviation which can be produced in a ray passing through the prism is $52^\circ\ 20'$, and find the minimum deviation.

9. Describe briefly the nature and properties of equipotential surfaces and tubes of force in an electrostatic field.

Two spherical conductors have unequal charges of opposite sign. Shew by general reasoning, without detailed calculations, that there will be either a point or a line of equilibrium. Indicate how the latter case depends upon the proximity of the spheres, the inequality of their charges, and also upon their dimensions. Draw rough diagrams of the lines of force and of the equipotential surfaces (in axial section) for each case.

10. State Ohm's law relating to the current in a conductor and the potential difference between the terminals.

A given electric current is divided between two conductors in parallel. Shew that the heat developed is less than if the current were divided in any other way than that which actually occurs.

FRIDAY, 31 *May*. 2—5.

1. Shew that the inverse of a circle with respect to a point is either a straight line or a circle. Shew also that a pair of inverse points with respect to the circle invert into a pair of inverse points with respect to the inverse circle.

If A', B' are the inverse points of A, B with respect to a circle and P is any point upon it, shew that the circles PAB, $PA'B'$ meet again upon the circle.

2. Prove that, between every pair of consecutive real roots of a rational integral algebraic equation $f(x) = 0$, there is an odd number of real roots of the equation $f'(x) = 0$.

Shew that the equation $x^5 - 5px^3 + 2q = 0$ will have three real roots, provided $27p^5 > q^2$.

3. Find the limiting value of $(\tan 2\alpha - 2\tan\alpha)/\alpha^3$, as α tends to zero.

Draw and discuss the graph of $\tan 3A \cot^3 A$, as A varies from 0 to $\frac{1}{2}\pi$, determining the maxima and minima, and the slope of the curve at $A = \frac{1}{2}\pi$.

4. An observer looking up the line of greatest slope of an inclined plane sees a tower due East of him. He walks 1000 feet up the plane in a direction 25° North of East, and, on reaching the level of the foot of the tower, finds that its elevation is 20° 25′. If the plane makes an angle of 10° with the horizontal, determine the height of the tower.

5. Differentiate $x^2 \sin(3 - 2x)$, and $\log_e \dfrac{1 - \sin x}{1 + \sin x}$ with respect to x.

Find the nth differential coefficient of $\dfrac{x^3}{(x-3)(x+1)}$.

6. Find expressions in terms of r and θ for the sine and cosine of the angle made by the tangent to a polar curve with the radius vector from the origin.

If an ellipse of eccentricity e is inverted with respect to its centre, the constant of inversion being k, prove that (s being measured along the arc of the inverse curve)

$$\frac{ds}{d\phi} = \frac{k^2}{a}\frac{(1 - e^2\cos^2\phi)^{\frac{1}{2}}}{1 - e^2\sin^2\phi},$$

where ϕ is the eccentric angle of the point on the ellipse corresponding to the point on the inverse curve, and a is the semi-axis major.

7. If $\int x^m (a + bx^n)^p\, dx = u_p$, prove that

$$(np + m + 1)\, u_p = x^{m+1}(a + bx^n)^p + anpu_{p-1}.$$

Prove also that

$$\int_0^\infty \frac{du}{\cosh^n u} = \frac{n-2}{n-1}\int_0^\infty \frac{du}{\cosh^{n-2} u} \quad (n > 2),$$

and find the value of $\displaystyle\int_0^\infty \frac{du}{\cosh^5 u}$.

8. Two lines $l_1x + m_1y = n_1$, $l_2x + m_2y = n_2$ are such that each passes through the pole of the other with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Prove that $a^2l_1l_2 + b^2m_1m_2 = n_1n_2$.

If the two lines above always pass through the ends of the major axis of the ellipse, shew that they intersect upon the ellipse $\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1$.

9. Four points are taken on a rectangular hyperbola $xy = c^2$. Find the condition that the chord joining two of the points shall be perpendicular to that joining the other two. Prove that this holds good for all three pairs of chords if it is true for one.

10. Find the equation of the polar plane of the point (x, y, z) with respect to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and prove that the polar planes of all points which lie on a straight line pass through another straight line (the polar line).

Shew that, if $a'^2(b^2 - c^2) + b'^2(c^2 - a^2) + c'^2(a^2 - b^2) = 0$, any normal to one of the ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1$$

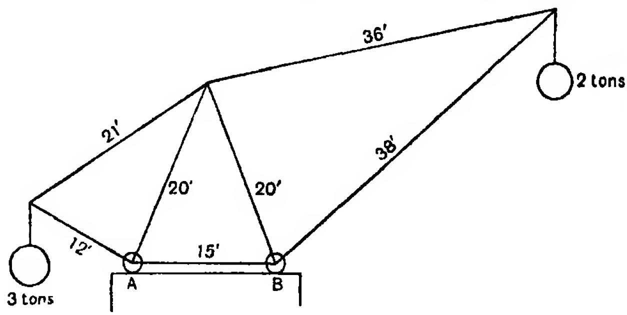
is perpendicular to its polar line with respect to the other.

SATURDAY, 1 *June*. 9—12.

1. Shew that any system of coplanar forces may be reduced to three forces acting along the sides of a triangle arbitrarily chosen in that plane, and shew how to find these three forces.

Apply your method to the following case. Forces indicated in magnitude and direction by $1, -2, 3, -4, 5, -6$ act in order along the sides of a regular hexagon. Reduce these forces to three, acting along the sides of the equilateral triangle having its base along the first side of the hexagon and its vertex in the centre of the opposite side of the hexagon.

2. Determine the stresses in each member of the given crane, loaded as shewn, and supported at A and B. The members are freely jointed at their extremities and their weights may be neglected. Their lengths are given in the figure. Distinguish between the members of the frame in thrust and in tension.



3. A lamina can turn freely about a fixed axis. Find the conditions that an impulsive force acting in the plane of the lamina may produce no instantaneous reaction at the axis.

A free lamina of any form is turning in its own plane about an instantaneous centre of rotation S. A point P in the line joining S to the centre of gravity G is brought to rest by an impulsive force passing through the point. Find the position of P in terms of SG and the radius of gyration about G, assuming that the velocity of G is the same as before, but reversed in direction.

4. A rigid ring hanging over a smooth peg is set spinning about its centre (which remains stationary) in its own vertical plane. It is completely fractured at one point A. Prove that the maximum bending moment experienced at the diametrically opposite point A' is

$$2r^3\omega^2 + gr^2(4+\pi^2)^{\frac{1}{2}},$$

and find the corresponding direction of AA'. The ring has unit mass per unit length, its radius is r and its angular velocity is ω.

5. The resistance to the motion of a train, for speeds between 20 and 30 miles per hour, may be taken as $\frac{1}{400}V^2 + 9$ in pounds weight per ton, where V is the velocity in miles per hour. Sketch a curve shewing how the horse-power per ton, necessary to

overcome the resistance, increases with the speed as the speed rises from 20 to 30 miles per hour, the train being on the level.

Steam is shut off when the speed is 30 miles per hour and the train slows down under the given resistance. In what time will the speed fall to 20 miles per hour?

6. Shew that the time of a small oscillation of a body about a horizontal axis perpendicular to the plane of motion is

$$2\pi\sqrt{\frac{k^2}{hg}},$$

where h is the distance of the centre of gravity of the body from the axis and k the radius of gyration of the body about the axis.

A flywheel weighing 5 tons is suspended from a pair of centres entering conical holes in the rim so that it can swing in a vertical plane. The line joining the centres is parallel to and distant 4 feet from the axis of the wheel, and the period of a complete swing is 3·2 seconds. Find the radius of gyration of the wheel about its axis, and determine how much energy the wheel would lose in falling from a speed of 120 to 90 revolutions per minute when revolving round its axis.

7. An object is placed in a square room whose walls are plane vertical mirrors. Shew that the four images, each of which is formed by reflexion in four mirrors in succession, are at the corners of a square, and that the orientation of each image is the same as that of the object.

8. Prove the formula

$$\frac{1}{u}+\frac{1}{v}=\frac{2}{r}$$

for reflexion at a spherical mirror, and explain the convention adopted with regard to the algebraic signs of the lengths involved.

A screen with a small hole is placed between a concave spherical mirror and its focus, so that the hole is on the axis of the mirror. Shew that the diameter of the part of the screen which will be visible by reflexion to an eye looking through the hole is $2d(1-u/r)$, where d is the diameter of the mirror, r its spherical radius and u the distance between it and the screen.

9. Find the potential of a sphere of radius a, having an electric charge Q uniformly distributed upon it.

An insulated sphere of radius 25 cm. is charged and afterwards connected to an electrometer by a long fine wire, the deflection being 75 divisions. The system is then joined to a distant insulated sphere of radius 12 cm. and the deflection falls to 53 divisions. Calculate the capacity of the electrometer.

10. State and prove the condition which must be satisfied by the resistances R_1, R_2, R_3, R_4 of the arms of a Wheatstone's Bridge, in order that no current may pass through the galvanometer.

If, after this condition has been satisfied, a new battery of electromotive force E be introduced into the galvanometer "bridge" so that the total resistance of the latter becomes R, find the current which will pass through the galvanometer.

SATURDAY, 1 *June*. 2—5.

1. Prove that the polar reciprocal of a circle with respect to an auxiliary circle is a conic, and indicate the positions of the foci and directrices.

Two parabolas with vertices A_1, A_2 have a common focus S. If P_1, P_2 are two points, one on each of the parabolas, such that the angles A_1SP_1, A_2SP_2 are equal and measured in the same sense, then the tangents at these points will meet on the common tangent of the two parabolas.

2. A circle cuts an ellipse in four points. Prove that the line joining two of the points and the line joining the other two points make equal angles with either axis.

Having given an ellipse, shew how to determine its axes.

3. State Maclaurin's theorem, and hence shew that the first five terms in the power series for $\log(1+\sin x)$ are

$$x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\frac{x^5}{24}.$$

4. Explain what is understood by the radius of curvature at any point of a curve, and shew how it may be found for the curve $y=f(x)$.

Find the curvature of the curve

$$2y = 2x^4 - 9x^3 + 11x^2$$

at the points $x = 0$, $x = 1$, and determine within what range of values of x the curve is concave to the axis of x.

5. Prove that

$$\text{(i)} \quad \int_0^{\frac{\pi}{2}} \frac{d\theta}{a^2 \cos^2\theta + b^2 \sin^2\theta} = \frac{\pi}{2ab},$$

$$\text{(ii)} \quad \int_0^{\frac{\pi}{4}} \frac{(\sin\theta + \cos\theta)\, d\theta}{9 + 16 \sin 2\theta} = \frac{1}{20} \log_e 3.$$

6. Shew that the kinetic energy of a rigid body in uniplanar motion is $\frac{1}{2} Mv^2 + \frac{1}{2} Mk^2 \omega^2$, where ω is the angular velocity of the body and v the velocity of its centre of gravity.

A circular plate of mass M and radius a has a mass m fixed in it at a distance b from the centre. An axis through the centre of the plate and perpendicular to it can slide without friction horizontally, while the plate revolves. If the plate is just disturbed from rest when m is in its highest position, find the angular velocities when the disc has made one-quarter and one-half a turn.

Determine the pressure on the axis in each case.

7. A cube of side $2a$ slides down a smooth plane inclined at an angle $2 \tan^{-1}\frac{1}{5}$ to the horizontal, and meets a fixed horizontal bar placed perpendicular to the plane of motion and at a perpendicular distance $\frac{a}{4}$ from the plane. Shew that, if the cube is to have sufficient velocity to surmount the obstacle when it reaches it, it must be allowed first to slide down the plane through a distance $\frac{107}{60}a$. The obstacle may be taken to be inelastic and so rough that the cube does not slip on it.

8. Prove that the electric intensity at a point just outside the surface of a conductor is $4\pi\sigma$, where σ is the surface density of electrification.

Prove also that the intensity at all distances on either side of a plane uniformly electrified with a charge σ per unit area is $2\pi\sigma$. How are these two results to be reconciled?

Shew that, of the total intensity $2\pi\sigma$ at a point A distant half an inch from the plane, half is due to the charge at points within an inch of A.

9. State and prove the conditions of equilibrium of a body floating in a heavy liquid.

A prism of square section floats in water with its long edges parallel to the water surface and the centre line of one of its faces hinged to an axis fixed in the surface of the water. Shew that if the specific gravity of the prism is $\frac{39}{64}$, the opposite face of the prism will be three-quarters immersed.

10. Prove that one of the principal points of a lens with one face plane is upon the curved surface, and determine the position of the other.

A thin convex lens of focal length f is placed in contact with the curved surface of a hemispherical lens of radius r, the two lenses being on the same axis. Prove that the focal length of the combination is $rf/\{r+(\mu-1)f\}$.

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WEDNESDAY, 28 *May*. 9—12.

1. Prove that opposite angles of a quadrilateral inscribed in a circle are supplementary.

A, B are two fixed points on a circle whose centre is O, P is any point on the circle and perpendiculars PM, PN are drawn to OA, OB. Prove that the length of MN is constant.

2. Prove the harmonic property of a quadrilateral.

From the intersection of one pair of common chords of two intersecting conics S, S' a tangent is drawn to the conic S. Prove that the tangent is divided harmonically by S'.

3. State the Binomial Theorem, and prove that

$$\left(\frac{1+x}{1-x}\right)^3 = 1 + 6x + 18x^2 + \dots + (4n^2+2)x^n + \dots .$$

4. Find the conditions that the expression $ax^2 + 2bx + c$ may be positive for all real values of x.

Prove that the expression

$$(a+c)(ax^2+2bx+c)-2(ac-b^2)(x^2+1)$$

has the same sign for all values of x, provided that the equation $ax^2+2bx+c=0$ has real roots.

5. Give a geometrical proof of the equation

$$\cos A+\cos B=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$

where neither A nor B exceeds $\frac{1}{2}\pi$.

Prove the formula

$$\cos^8 A-\sin^8 A=\cos 2A-\tfrac{1}{4}\sin 2A\sin 4A.$$

Eliminate A between the equations

$$\cos A+\sin A=a,\qquad \cos 2A=a'.$$

6. A weight is hung from an endless string of length l passing over a pulley of 1 foot diameter. The point of suspension of the weight from the string is at distance d below the centre of the pulley. Shew that $l=\pi-\cos^{-1}\dfrac{1}{2d}+(4d^2-1)^{\frac{1}{2}}$. Draw a curve to shew the relation between l and d for values of d between 1 and 3 feet, and determine for what value of d the value of l is 5 feet.

7. If $f(x)$ is a rational integral function of x, prove that between any two real roots of the equation $f(x)=0$ there lies one root at least, of the equation $f'(x)=0$. Hence shew how to use $f'(x)$ to find any multiple roots of the former equation, and use the method to determine the factors of the expression

$$x^5-3x^4-6x^3+10x^2+21x+9.$$

8. Integrate

$$\int\frac{(x-1)\,dx}{x\sqrt{(x^2-9)}},\qquad \int\frac{dx}{\sin^{\frac{1}{2}}x\cos^{\frac{5}{2}}x},\qquad \int\frac{(2-\sin x)\,dx}{\sin x\,(1-\cos x)}.$$

9. Shew that the length of an arc of the curve whose equation is $x=f(t)$, $y=\phi(t)$, where t is a parameter, is

$$\int\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}^{\frac{1}{2}}dt.$$

Hence find the length of an arc of the cycloid

$$x = a(\theta + \sin\theta), \quad y = a(1 + \cos\theta)$$

and determine the centre of gravity of a uniform wire in the shape of the arc of the cycloid between $\theta = \pi$ and $\theta = -\pi$.

10. Prove the formula $\cos\theta = ll' + mm' + nn'$, for the cosine of the angle between two lines whose direction cosines are (l, m, n), (l', m', n').

A cube rests on a plane inclined at an angle α to the horizon, one diagonal of the base lying along a line of greatest slope. Find the inclinations of the diagonals and faces of the cube to the horizon.

WEDNESDAY, 28 *May*. $1\frac{1}{2}$—$4\frac{1}{2}$.

1. Prove that any transversal cuts the rays of a pencil formed by four concurrent lines in a range whose cross ratio is constant.

Prove also that any plane cuts four collinear planes in a pencil whose cross ratio is constant.

2. Shew how to determine, when possible, a point on either of two non-intersecting straight lines which shall be at a given distance from the other line. How many solutions of the problem will there be, and what is the minimum given distance in order that the problem may be possible?

3. Prove that, if α, β are two roots of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

the other two roots will be equal if

$$3(\alpha^2 + \beta^2) + 2\alpha\beta + 2p(\alpha + \beta) + 4q - p^2 = 0.$$

4. Write down the expansions of $\sin x$ and $\cos x$ in ascending powers of x.

Prove that, if l is the length of the chord of a circular arc, l_1 the chord of two-thirds of the arc and l_2 the chord of one-third of the arc, the length of the arc is approximately

$$\tfrac{1}{10}(l - 9l_1 + 45l_2)$$

and that this approximation is correct to the order θ^6 where θ is the angle which the arc subtends at its centre.

5. Prove that in any triangle ABC

$$\tan \tfrac{1}{2}(B-C) = \frac{b-c}{b+c}\cot \tfrac{1}{2}A.$$

Find, correct to one foot, the third side of a triangle in which the lengths of two sides, including an angle 103° 52′, are 417 and 259 feet respectively.

6. A man starts up the line of greatest slope of an inclined plane and after walking 1000 yards finds that he has risen 150 feet. He turns to the right and after going 1000 yards has risen a further 100 feet. If the line of greatest slope is due North, determine (i) the bearing to the North of the new path, (ii) the bearing of the original starting point as seen from the last position.

7. Find the length of the perpendicular from the point (x', y') upon the line $ax + by + c = 0$.

Find the coordinates of the in-centre of the triangle formed by the lines

$$x + y + 11 = 0, \quad x - y + 1 = 0, \quad x - 7y + 7 = 0.$$

8. State and prove an expression for the area of a triangle of which the corners are (x_1, y_1), (x_2, y_2), (x_3, y_3).

Three normals having the equations $y = m_r x + c$, $(r = 1, 2, 3)$ are drawn to the parabola $y^2 = 4ax$. Prove that the area of the triangular space enclosed by them is

$$\tfrac{1}{2}a^2 (m_1 \sim m_2)(m_2 \sim m_3)(m_3 \sim m_1)(m_1 + m_2 + m_3)^2.$$

9. Prove that there are two real planes passing through the centre of an ellipsoid and intersecting the surface in circles.

Normals are drawn at all points of the circular central sections of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (a > b > c).$$

Prove that they intersect the plane $x = 0$ upon the ellipse

$$\frac{b^2 y^2}{a^2 - b^2} + \frac{c^2 z^2}{a^2 - c^2} = a^2 - b^2.$$

10. Solve the equations

$$\text{(i)} \quad (x+y-2)\frac{dy}{dx} = x - y + 3.$$

$$\text{(ii)} \quad \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = e^{-2t}\sin 4t.$$

FRIDAY, 30 *May*. 9—12.

1. A uniform rod rests with its ends on two smooth planes inclined at 30° and 45° respectively to the horizontal. Prove that the inclination of the rod will be $\cot^{-1}(\sqrt{3}+1)$. Also find what weight, fixed to the rod at a quarter of its length from one end, will suffice to enable the rod to rest horizontally.

2. Two spheres, whose radii are a_1, a_2, rest inside a smooth hollow vertical right cylinder, of which the external radius is c, and the internal radius b, where $b < a_1 + a_2$. Prove that, if the sphere a_1 is the lower, the cylinder will not overturn if its weight exceeds

$$\frac{w}{c}(2b - a_1 - a_2),$$

where w is the weight of the upper sphere.

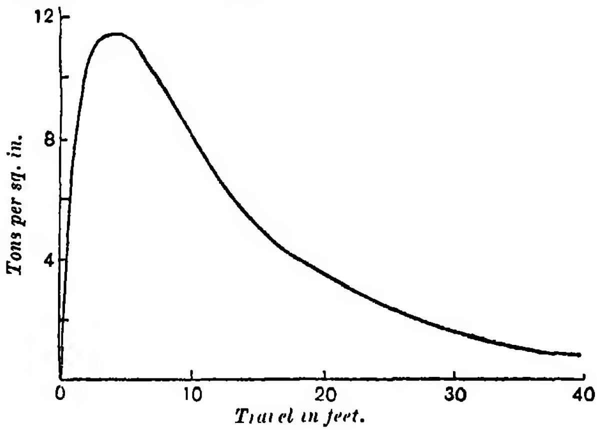
3. A suspension chain carries a load uniformly distributed on a horizontal platform. Shew that if the weight of the chain may be neglected the curve of the chain is a parabola.

In a suspension bridge of 400 feet span and 40 feet dip the whole weight supported by the two chains is 2 tons per horizontal foot. Find the horizontal tension in the chains and the tension at the points of support.

If a simple telegraph wire has a span of 75 yards and a sag in the middle of 1 foot, shew that the tension in the wire is approximately 480 pounds weight, where the weight of the wire is 400 pounds per mile.

4. A uniform chain 30 centimetres long, having a mass of 1 gramme per centimetre, lies partly in a straight line along a rough horizontal table, perpendicular to the edge. The portion hanging over the edge is just sufficient to cause the chain to commence to slip. The coefficient of friction with the table being $\frac{1}{2}$, find the velocity of the chain and its tension at the edge of the table when x centimetres have slipped off.

5. The diagram shews the variation of the pressure along the bore of a gun 40 feet in length. Determine the curve of velocity of the shot along the length of the gun, and the approximate time taken to reach the end, neglecting the frictional resistance to motion, the recoil, and the energy of rotation of the shot. The diameter of the bore is 6 inches and the weight of the shot 100 pounds.



6. State Kepler's three laws of planetary motion about the sun, and prove the "theorem of equal areas." Indicate what modification is necessary when the finite mass of the sun is taken into account.

The greatest and least velocities of a certain planet in its orbit round the sun, which may be regarded as fixed, are 30·0 and 29·2 kilometres per second respectively. Find the eccentricity of the orbit.

7. Find the centre of pressure of a rectangle immersed in a uniform liquid and having one of its sides in the surface.

A cube, with edges of length $2a$, is immersed in a liquid and has one edge in the surface and the two faces through that edge equally inclined to the horizontal. Find the centres of pressure of all the faces.

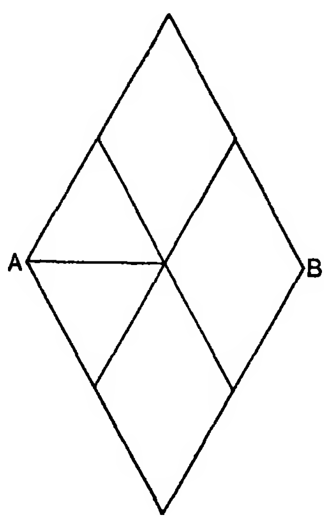
8. Describe some optical method by means of which the focal length of a concave mirror may be determined.

An object has a virtual image three times its size in a concave mirror of focal length 12 centimetres. Determine the position of the object. Is the image erect or inverted?

9. Electric charges $e_1, e_2, \ldots$ are placed at points $O_1, O_2, \ldots$ along a straight line. Prove that at all points P along any one line of force due to the system the value of $\Sigma e_n \cos\theta_n$ is constant, where θ_n is the angle between the line of charges and PO_n $(n = 1, 2, \ldots)$.

Positive and negative unit charges are situated at two points A, B. Find at what distance from AB the plane normal to and bisecting AB is cut by the lines of force which issue from A in a direction parallel to this plane.

10. A network of wires as shewn is composed of 13 elements all of equal length and equal resistance r. A current I is led in at A and out at B. Indicate its distribution during its passage through the network, and prove that the equivalent resistance between A and B is $\frac{6}{5}r$.



FRIDAY, 30 *May*. $1\frac{1}{2}$—$4\frac{1}{2}$.

1. Prove that the circle circumscribing the triangle formed by three tangents to a parabola passes through the focus: and that no new result is obtained by reciprocating this theorem with respect to the focus.

2. Experiments are made upon two related quantities x and y, and corresponding values are observed. Explain how it may be determined whether either

$$\text{(i) } y = ax^n, \text{ or (ii) } y = \frac{ax}{x+b}$$

approximately expresses the relation between them.

Water is discharged over a weir and it is found that for different heights (h) of the free surface of the water above the bottom of the weir, the discharge (Q) is given by the following table:

h	4	6	8	10	12
Q	650	1740	3640	6360	9790

Shew that the relation between Q and h is of the form $Q = ah^n$, and determine the best values of a and n.

3. Draw a graph of the function

$$\sec^4\theta - 6\sec^2\theta + 8$$

for values of θ between 0 and π, and find by exact methods the values of θ within this range for which the function vanishes. Shew also that the minimum value of the function is -1.

4. Shew that if R, r are the radii of the circumscribed and inscribed circles of a triangle,

$$R = \frac{abc}{4S}, \quad r = \tfrac{1}{2}(b+c-a)\tan\frac{A}{2}.$$

$ABCD$ is a quadrilateral inscribed in a circle: shew that if $AB + CD = BC + DA$,

(i) a circle can be drawn touching the four sides,

(ii) the circles inscribed in the triangles ABD, BCD touch one another,

(iii) $\tan\frac{A}{2} = \sqrt{\frac{BC \cdot CD}{DA \cdot AB}}$.

5. What is the geometrical meaning of

$$\frac{d}{dx}\phi(kx) = k\phi'(kx)?$$

Prove that, if $y = \frac{f_1 f_2 \dots f_m}{g_1 g_2 \dots g_n}$, where $f_1, g_1, f_2, g_2, \dots$ are functions of x,

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{f_1}\frac{df_1}{dx} + \frac{1}{f_2}\frac{df_2}{dx} + \dots\right) - \left(\frac{1}{g_1}\frac{dg_1}{dx} + \frac{1}{g_2}\frac{dg_2}{dx} + \dots\right).$$

Differentiate $\sin^{-1}\{2ax\sqrt{(1-a^2x^2)}\}$, $\dfrac{(1+x^2)^{\frac{1}{2}} + (1-x^2)^{\frac{1}{2}}}{(1+x^2)^{\frac{1}{2}} - (1-x^2)^{\frac{1}{2}}}$.

6. Prove that the coordinates of the centre of curvature C at any point (x, y) of the curve $y^2 = \frac{x^3}{a}$ are

$$\left(-x - \frac{9}{2}\frac{x^2}{a},\ 4y + \frac{4}{3}\frac{ay}{x}\right).$$

Draw a rough diagram of the original curve and of the locus of C, and prove that the radius of curvature of the *locus* at the origin is $\frac{8}{3}a$.

7. Trace the curve $r\cos\theta = a\sin 3\theta$, and shew that the area of a loop is $\frac{1}{4}a^2(9\sqrt{3} - 4\pi)$.

8. The coordinates of any two points P, Q are (x_1, y_1), (x_2, y_2), the origin being at O. Circles are described on OP, OQ as diameters. Prove that the length of their common chord is $(x_1 y_2 \sim x_2 y_1)/PQ$.

9. Find the equation of the tangent to the ellipse

$$x^2/a^2 + y^2/b^2 = 1,$$

at the point (x', y').

Prove that the normals to the ellipse at the points where it is met by the line

$$\frac{x}{a\cos\alpha} + \frac{y}{b\sin\alpha} = 1,$$

meet in the point

$$x = -\frac{a^2-b^2}{a}\cos^3\alpha,\quad y = \frac{a^2-b^2}{b}\sin^3\alpha.$$

10. Prove that, if the coordinate axes be suitably chosen, *the equations of any two straight lines may be written in the* form

$$y = x\tan\alpha,\quad z = c\,;\quad y = -x\tan\alpha,\quad z = -c.$$

Shew that, through any point (ξ, η, ζ), it is possible to draw a line which will intersect each of these two lines, and that the line so drawn will meet the plane $x = 0$ in the point

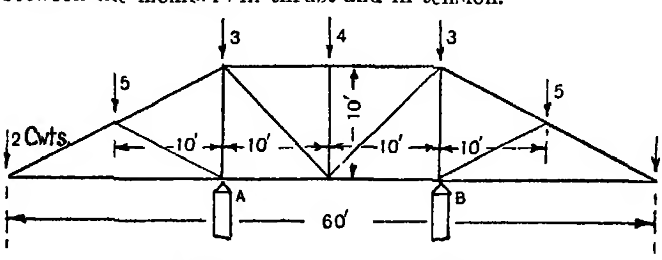
$$y = c\,(\eta^2 - \xi^2\tan^2\alpha)/(c\eta - \xi\zeta\tan\alpha),$$
$$z = c\,(\eta\zeta - c\xi\tan\alpha)/(c\eta - \xi\zeta\tan\alpha).$$

SATURDAY, 31 *May*. 9—12.

1. *Explain the terms shearing force and bending moment at* a section of a beam under given vertical loads.

A beam AD, 35 feet long and supported at points B and C distant 10 and 5 feet respectively from the ends A and D, carries a load 10 tons uniformly distributed along AB and a load of 5 tons at D. If the weight of the beam is neglected draw diagrams shewing the shearing force and bending moment at all points of the beam.

2. Determine the stresses in each member of the given station roof frame supported at A and B and symmetrically loaded as shewn. The members are freely jointed at their extremities and their weights may be neglected. Distinguish between the members in thrust and in tension.



3. A mass of M pounds oscillates vertically at the end of an elastic string with amplitude $\frac{M}{m}$ feet, where m pounds is the weight which stretches the string one foot. If, when at the

lowest point of its travel, M receives a downward impulse B in poundal second units, shew that the time in seconds from the lowest point of the subsequent travel of M to its highest is

$$\frac{B}{Mg} + \left(\pi - \tan^{-1} \frac{B}{M}\sqrt{\frac{m}{Mg}}\right)\sqrt{\frac{M}{mg}}.$$

4. A machine gun of mass M stands on a horizontal plane and contains shot of mass M'. The shot is fired at the rate of mass m per unit of time with velocity u relative to the ground If the coefficient of sliding friction between the gun and the plane is μ, shew that the velocity of the gun backward by the time the mass M' is fired is

$$\frac{M'}{M}u - \frac{(M+M')^2 - M^2}{2mM}\mu g.$$

5. A locomotive drawing a total weight of 264 tons on the level is exerting a tractive force of 20000 pounds weight at the speed of 15 miles per hour. It works at constant horse-power until its speed is 60 miles per hour, when it is just able to overcome the resistances to motion, which may be taken to vary as the square of the velocity. Shew that it reaches the speed of 45 miles per hour from the speed of 15 miles per hour in a distance of approximately 5080 feet.

6. A particle is projected from a point A so as to pass through a given point B and to return to the same level as A at a point C. Shew that the velocity with which the particle must be projected is

$$\sqrt{\frac{AN^2 \cdot NC^2 + NB^2 \cdot AC^2}{2AN \cdot NB \cdot NC} \cdot g},$$

where N lies in AC directly below B.

7. State the laws of refraction of light.

A ray of light is incident on the curved surface of a glass hemisphere in a direction parallel to its axis. The radius of the hemisphere is 2 inches and its refractive index 1·5. Prove that the ray will not pass through the hemisphere if its distance from the axis is greater than 1·987 inches.

8. Prove the formula $\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}$, for the refraction at a spherical surface.

The radii of curvature of the front and back faces of a thin double convex lens are r, s, and a small object is placed upon the axis at a distance x from the lens. Find the position of the image formed by two refractions at the front face and a single reflexion at the back face. Find also the magnification produced.

9. Shew how to find the electrostatic field due to a conducting uninsulated sphere in the presence of a given point charge.

A point charge e is placed at a distance f from the centre of an insulated uncharged sphere of radius a. Shew that the total charge on the smaller part of the sphere cut off by the polar plane of the point is

$$-\frac{1}{2}e\left\{1+\frac{a^2}{f^2}-\sqrt{\left(1-\frac{a^2}{f^2}\right)}\right\}.$$

10. Find the least number of Grove's cells, each having an E.M.F. of 1·87 volts and a resistance of 0·17 ohm, which will send a current of 16 amperes through a resistance of 1·5 ohms, and shew how the cells should be arranged.

SATURDAY, 31 *May*. $1\frac{1}{2}$—$4\frac{1}{2}$.

1. Prove that two circles of a coaxal system can be drawn so as to touch a given line.

If A, B are the points of contact of the circles with the line, prove that (1) the circle on AB as diameter passes through the limiting points, (2) the polar of A with respect to every circle of the system passes through B.

2. A parallelogram is inscribed in an ellipse. Shew that its sides are parallel to conjugate diameters, and find its greatest area.

Two chords OPQ, RPS of an ellipse are drawn parallel to two fixed lines in the plane. Prove that the ratio of the rectangles contained by their segments is independent of the position of P.

3. Determine the point of inflexion upon the curve

$$y = 2x^3 - 3x^2 - 12x + 5,$$

and find within what limits the curve is (i) concave, (ii) convex

towards the positive direction of the axis of y. Trace the curve and determine the abscissae of the points at which the tangent to the curve is parallel to $3y = 20x + 5$.

4. Shew how to determine the change in a function of several variables due to small changes in the variables.

Francis' formula for the number of cubic feet of water Q passing per minute over a weir of breadth b and depth h is $Q = 3{\cdot}33\,(b - 0{\cdot}2h)\,h^{\frac{3}{2}}$: if the breadth of the weir is measured to be 8 feet and the depth 1·25 feet, determine the maximum percentage error made in estimating the quantity Q due to errors of 1 inch in the measurement of both the breadth and the depth of the weir.

5. Explain the method of integration by parts.

Integrate $\int e^{ax} \cos (bx + c)\, dx$, $\int e^{ax} \cos^2 (bx + c)\, dx$.

6. A uniform straight heavy rod AB of mass M is freely jointed about a smooth horizontal axis at A, and is supported at an inclination θ to the vertical by a light string which is perpendicular to the rod and attached to it at B. The string is suddenly cut. Find the pressures on the axis at A before and immediately afterwards.

7. A uniform trap door swinging about a horizontal hinge is closed by a spring coiled about the hinge. The spring is coiled so that it is just able to hold the door shut in the horizontal position. The horizontal opening which the door closes is in a body which is mounting with uniform acceleration f. Shew that, if $f = \left(0{\cdot}57 + \dfrac{1{\cdot}23}{\alpha}\right) g$, the door starting from the vertical position will just reach the horizontal position, α being the angle through which the spring is coiled when the door is in the horizontal position.

8. Explain the value of equipotential surfaces and lines of force as indicators of the direction and relative intensity at each point of an electrostatic field.

Two equal plane plates of any shape are placed opposite and parallel to one another at a distance $2d$ apart. One plate being earthed, a charge E placed on the other raises its potential to

ii. Prove that either plate alone, placed at distance d from a parallel infinite conducting plane at zero potential, would be raised to potential $\frac{1}{2}V$ by the same charge E.

9. Two boards of the same length and the same small thickness and of widths a, b are fixed together lengthwise along a common edge with their planes at right angles. Shew that they will float in water with the common edge downwards and with the board of width a inclined at an angle θ to the horizon, given by

$$\frac{\tan^3\theta - 1}{(1+\tan\theta)^2} = \frac{b^2\tan\theta - a^2}{\rho(a+b)^2},$$

where ρ is the specific gravity of the boards.

10. Find the principal foci and principal points of a lens in which the radii of curvature of the faces are r, s and the thickness is t.

A cylindrical block of glass of refractive index 1·5 has spherical convex ends whose radii are $\frac{1}{2}$ inch and $2\frac{1}{2}$ inches, and the length along the axis is 9 inches. Prove that the magnification produced in all small objects along the axis, on the side next to the end with the greater curvature, is equal to -5, and that the image of an object at a distance of $1\frac{5}{8}$ inches from this end coincides in position with the object.

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THURSDAY, 3 *June*. 9—12.

1. $ABCDE$ is a regular pentagon and AC, BE intersect in H. Prove that $AB = CH - EH$ and that AB is a tangent to the circle BHC.

ii. State and prove the harmonic property of a pole and polar for a circle.

ACA' and BCB' are conjugate diameters of an ellipse and O is any point on AA'. Any line parallel to BB' cuts AA' in N and any pair of conjugate lines through O in P and Q. Prove that

$$PN \,.\, NQ : ON^2 = BC^2 : AO \,.\, OA'.$$

3. Prove that as x varies from $-\infty$ to $+\infty$ the function

$$y = \frac{(x-\alpha)^2}{x-\beta}$$

assumes twice over all values except those in an interval of length $4\,|\alpha - \beta|$, and locate this range of values precisely.

4. Prove that the multiple roots of the equation

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

are all the common roots of

$$f(x) = 0, \quad f'(x) = 0.$$

If the a's are integers, and there is only one multiple root, shew that this root must be real and rational, whatever its degree of multiplicity.

Determine by inspection of the equations

$$f(x) = 0, \quad f'(x) = 0, \ \ldots$$

the multiple roots of

$$x^5 - x^4 - 4x^2 + 7x - 3 = 0.$$

v. Find, correct to about a minute, the values of x between $0°$ and $360°$ which satisfy the equation

$$5 \sin x - 7 \cos x = 6.$$

Solve completely the equation $\sin 4x + \cos 3x = 0$, and prove that $\sin 4x + \cos 3x$ can be expressed in the form

$$8 \cos x\,(\sin\alpha - \sin x)(\sin\beta - \sin x)(\sin\gamma - \sin x),$$

determining suitable values for α, β, γ.

vi. Prove that, if

$$\phi(x) = x - \tfrac{1}{3}\left(8 \sin \tfrac{1}{2}x - \sin x\right), \quad \phi'(x) = \tfrac{8}{3} \sin^4 \tfrac{1}{4}x,$$

and deduce that

$$\frac{32}{15} \sin^5 \frac{x}{4} < \phi(x) < \frac{32}{15}\left(\frac{x}{4}\right)^5.$$

By taking $x = \frac{1}{6}\pi$, prove that $4(\sqrt{6} - \sqrt{2}) - 1$ is an approximate value of π, and use the above inequalities to form a rough estimate of the error.

7. Integrate

$$\int \frac{dx}{(x-\lambda)\sqrt{x-\mu}}, \quad \int \frac{x^3\,dx}{x^2+\lambda^2}, \quad \int \frac{dx}{(x+1)(x+2)(x+3)}.$$

8. Prove that the area enclosed between the curve $r=f(\theta)$ and the radii vectores $\theta=\theta_1, \theta=\theta_2$ is

$$\tfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta.$$

The area of the loop of the curve $(x^2+y^2)^2-4axy^2=0$ in the positive quadrant is $\frac{1}{4}\pi a^2$.

ix. The equations of the sides of a triangle are $u=0$, $v=0$, $w=0$, where

$$u=x\cos\alpha+y\sin\alpha-p,\quad v=x\cos\beta+y\sin\beta-q,$$
$$w=x\cos\gamma+y\sin\gamma-r.$$

Prove that the perpendiculars from the corners on the opposite sides meet in a point (the orthocentre) given by

$$u\cos(\beta-\gamma)=v\cos(\gamma-\alpha)=w\cos(\alpha-\beta).$$

Three points (x', y'), (x'', y''), (x''', y''') lie on the rectangular hyperbola $xy=a^2$. Prove that the orthocentre (x, y) of the triangle formed by the points is given by

$$xx'x''x'''=yy'y''y'''=-a^4,$$

and lies on the hyperbola.

x. Prove that the equation

$$x^2+y^2+z^2+2fx+2gy+2hz+c=0$$

represents a sphere if $f^2+g^2+h^2>c$.

Prove that a sphere can be drawn to pass through the midpoints of the edges of the tetrahedron whose faces are $x=0$, $y=0$, $z=0$, $x/a+y/b+z/c=2$.

Thursday, 3 *June*. $1\frac{1}{2}$—$4\frac{1}{2}$.

1. Given the limiting points A, B of a system of coaxal circles, construct geometrically the circle of the system which passes through a given point P.

Prove that, if P moves on a fixed straight line and the coaxal circle through P cuts AP in Q, the locus of Q is a straight line.

2. A regular polygon of 20 sides is inscribed in a circle of radius 10 inches. Find the difference between the perimeters of the circle and the polygon.

Find the greatest number of circular discs, of diameter 1 inch, which can be placed, without overlapping, between two concentric circles of radii 4 and 5 inches.

3. Expand the function $\dfrac{3 \sin x}{2 + \cos x}$

in powers of x up to and including x^5.

Give a rough sketch of its graph, finding its maximum and minimum ordinates, its points of inflexion and the slope of the graph at these points. Shew that certain portions are nearly rectilinear.

4. A, B are two points, a ft. apart, on a horizontal plane in line with the base C of a tower. The elevations α, β of the top of the tower are observed at A and B, $(\beta > \alpha)$. Find the distance x of C from the middle point of AB.

Prove that the increment δx in x due to small increments $\delta\alpha$, $\delta\beta$ in α, β is given by

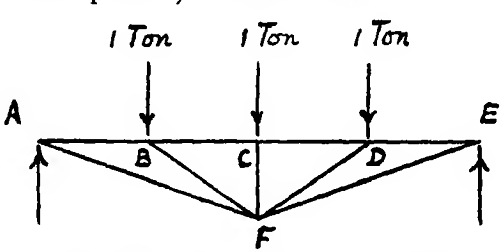
$$\frac{\delta x}{x} = \frac{\sin 2\beta\,\delta\alpha - \sin 2\alpha\,\delta\beta}{\sin(\beta+\alpha)\sin(\beta-\alpha)}.$$

Shew that, if the uncertainty in a measured angle is $\pm 6'$ and if $\beta - \alpha = 20^\circ$, the uncertainty in x will not exceed 1 per cent.

5. Find the equation of the tangent to the parabola $y^2 = 4ax$ which is parallel to the line $y = mx$.

Find the equation of the common tangent to the parabolas $y^2 = 4ax$, $2x^2 = ay$.

vi. The diagram shews a framework loaded and supported as shewn. Neglecting the weights of the members, and assuming smooth hinge joints at their extremities, determine the stresses (tension or compression) in each member.



$AB = BC = CD = DE$, $AC = 3 \,.\, CF$.

vii. Defining Simple Harmonic Motion by an equation of the type $y = a \cos pt$, shew that the acceleration in such a motion is proportional to the displacement and directed towards the mean position.

A string AB consists of two portions AC, CB, of unequal lengths and elasticities. The composite string is stretched and held in a vertical position with the ends A and B secured. A particle is attached to C, and the steady displacement of C is found to be δ. Shew that a further small vertical displacement of C will cause the particle to execute a Simple Harmonic Motion and that the length of the simple equivalent pendulum is δ. Both portions of the string are assumed to be in tension throughout, and the weight of the string may be neglected.

viii. The lock of a railway carriage door will only engage if the angular velocity of the closing door exceeds ω. The door swings about vertical hinges and has a radius of gyration k about a vertical axis through the hinges, whilst the centre of gravity of the door is at distance a from the line of hinges. Shew that if the door be initially at rest and at right angles to *the side of the train, which then commences to move with* uniform acceleration f, the door will not close unassisted unless $f > \frac{1}{2}\omega^2 k^2/a$.

ix. Shew that the depth of the Centre of Pressure of a lamina totally immersed in water with its plane vertical is greater than the depth z of the Centre of Gravity below the free surface of the water, by an amount which varies inversely as z; atmospheric pressure is neglected.

A circular lamina of radius 1 foot is totally immersed in water with a horizontal diameter fixed at a depth of 3 feet. Shew that if the lamina be rotated about this diameter, the Centre of Pressure always lies on a fixed vertical circle of diameter 1 inch.

x. The substitution $\mu = -1$ converts formulae for refraction into the corresponding formulae for reflection. Prove this statement.

A thin double convex lens ($\mu = 1{\cdot}5$) has surfaces of radii $4{\cdot}5$ and 3 inches. Coaxial with it, and touching it, is a convex spherical mirror of radius $3{\cdot}6$ inches. Shew that the combination behaves approximately like a plane mirror for rays near the axis of the combination.

FRIDAY, 4 *June*. 9—12.

1. Prove that the two tangents drawn to a central conic from an external point are equally inclined to the focal distances of that point. What does this theorem reduce to in the case of the parabola?

PT, $P'T$ are a pair of tangents to an ellipse and PO, $P'O$ the corresponding normals. Prove that if PP' passes through one focus, OT passes through the other.

2. Let $N = x^n(1+\epsilon)$, where ϵ is a small number, so that x is an approximate value of $N^{\frac{1}{n}}$. Shew that if

$$x_1 = x \cdot \frac{(n-1)x^n + (n+1)N}{(n+1)x^n + (n-1)N},$$

then x_1 is a nearer approximation to $N^{\frac{1}{n}}$, being correct up to and including terms in ϵ^2.

Taking $x = 1{\cdot}4$, find an approximate value of $\sqrt{2}$.

3. According to Van der Waals, the relation between the pressure (p), volume (V), absolute temperature (T), of a given mass of a gas is given by

$$\left(p + \frac{a}{V^2}\right)(V-b) = RT,$$

where a, b, and R are constants. Regarding this relation as a cubic equation in V, shew that the roots will be all equal, provided

$$p = a/27b^2, \quad T = 8a/27Rb,$$

and that the single value of V then obtained will be $3b$.

For Carbon Dioxide, taking $273R = 1{\cdot}00646$, $a = {\cdot}00874$, $b = {\cdot}0023$, shew that the above value of T becomes about $305{\cdot}4$.

4. Explain the method of integration by parts.

Integrate $\int x^3 \cos ax\,dx$. Why does the method fail to give a complete result when applied to

$$\int \frac{\sin ax}{x^3}\,dx?$$

5. Integrate $$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{-\frac{1}{2}x};$$

give the particular integral for which

$$y = 1, \quad \frac{dy}{dx} = -\tfrac{1}{2} \text{ when } x = 0.$$

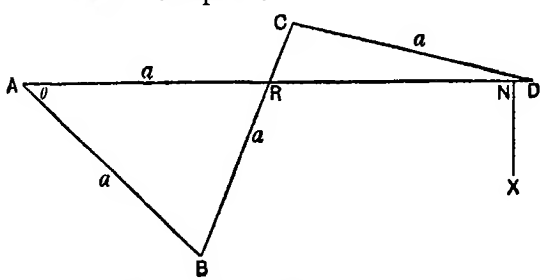
Integrate
$$x\frac{dy}{dx} + y = x^3y^2.$$

vi. A string stretched over a rough plane curve is just on the point of slipping. Shew how to find the ratio of the tensions at its extremities.

A belt-driven pulley of diameter 2 feet transmits 10 Horse-Power when running at 240 revolutions per minute. Shew that if the belt is just on the point of slipping, and subtends an arc of 180° of the pulley, the biggest tension in the belt is 306 lbs. nearly. The coefficient of friction between belt and pulley can be taken as 0·4.

(1 Horse-Power = 33,000 ft.-lbs. per minute.)

vii. Prove that a rigid lamina moving in its own plane in any manner has in general an instantaneous centre of rotation. What is the case of exception?



Three equal rods AB, BC, CD, are hinged together and supported as in the figure by a fixed hinge at A and a fixed ring at R through which BC slides, while D can slide along AR produced. Draw a sketch to shew the instantaneous centres of BC and CD at any moment, and prove that, if $AR = a$, as θ tends to 60° the instantaneous centre of CD tends to X, where $AN = 2a$ and $NX = a/\sqrt{3}$.

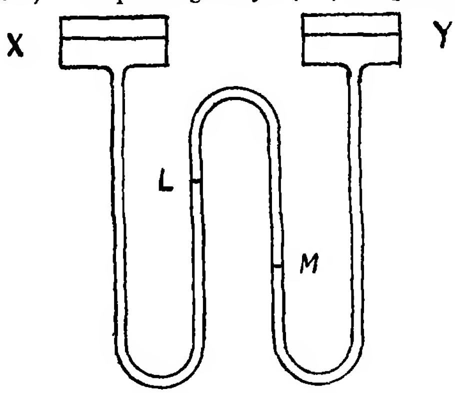
viii. Two rough uniform solid circular cylinders A and B of weights w_1 and w_2 respectively lie in contact with their axes horizontal, on a perfectly rough inclined plane. B is higher than A, whilst the radius of B is greater than that of A. If

the angle of friction for A on B be ϕ, and θ be the angle which the plane containing the cylinder axes makes with the inclined plane, shew that the cylinders will just roll down the inclined plane if

$$w_2 = w_1 \frac{\sin\phi + \cos(\theta+\phi)}{\sin\phi - \cos(\theta+\phi)}.$$

The angle which the inclined plane makes with the horizontal may be taken as less than $90° - \theta$.

ix. The diagram shews an arrangement for measuring small differences of air pressure. Except at its ends, the tube is of uniform internal cross section a. At the ends, the cross sectional areas are A (at X), B (at Y). Water occupies the tube from X to L, oil of specific gravity σ (<1) occupies the portion



L to M, whilst the remaining part of the tube is filled with water. Shew that the effect of producing a small difference of pressure due to H inches of water between the air acting on X and that acting on Y, is to cause a change in the difference of level between L and M amounting to

$$2H \Big/ \left\{a\left(\frac{1}{A} + \frac{1}{B}\right) + 2(1-\sigma)\right\}.$$

x. Prove Gauss' Theorem that

$$\int E_n \, dS = 4\pi \Sigma e,$$

where the integral is taken over any closed surface S, E_n is the

component of the electric force along the outward normal to S, and Σe is the sum of the electric charges inside the surface S.

Hence or otherwise shew that the electric force due to a freely charged sphere is the same at all external points as if the charge were collected at the centre of the sphere.

FRIDAY, 4 *June*. $1\frac{1}{2}$—$4\frac{1}{2}$.

1. Prove that any conic can be projected into a circle having the projection of any given point for centre.

An ellipse, whose principal axes are ACA' and BCB', is projected from a point O in the plane through C perpendicular to BB', and the vanishing line is parallel to BB' and cuts AA' in N. Prove that the centre projects into a focus if

$$ON : CN = BC : AC,$$

and that, if NB cuts the ellipse in P and L is the foot of the perpendicular from P on AA', L projects into the second focus.

2. Express $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ in terms of the sides of a triangle ABC and deduce that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{abc}.$$

How many independent relations exist connecting the angles of the triangles which form the faces of a tetrahedron $ABCD$? Shew that one of these is

$$\sin DBC \sin DCA \sin DAB = \sin DCB \sin DAC \sin DBA.$$

3. Apply Taylor's Theorem to expand $\log_{10}(a+x)$ in powers of x.

A table of logarithms to base 10 is given to 7 places of decimals for all numbers of three digits. Explain how to find the logarithm of a number of five digits by simple interpolation, and examine the degree of accuracy of the result thus found.

4. Find the length of the perpendicular from the point (x''', y''') on the line joining the points (x', y'), (x'', y''), and deduce the area of the triangle formed by the points.

Shew how to determine the circle which passes through the points $(0, 0)$, $(0, 2a)$ and touches the line $x\cos\alpha + y\sin\alpha = p$, proving that two such circles exist unless the line passes between the points.

5 Find the equation of a plane which passes through a given point and is parallel to each of two given lines

Three points A, B, C on a horizontal plane are chosen so that BAC is a right angle and $AB = AC = a$ Vertical borings at A, B, C strike coal at depths h_1, h_2, h_3 Assuming that the upper surface of the coal is a plane, prove that this plane dips at an angle θ to the horizontal such that

$$a \tan\theta = \{(h_2 - h_1)^2 + (h_3 - h_1)^2\}^{\frac{1}{2}},$$

and find the line in which this plane meets the plane ABC

vi A *torpedo which drives itself through the water at a* constant speed V, its axis preserving in all circumstances an invariable direction is fired from O at an enemy ship P whose constant speed through the water is v At the moment of firing the direction of motion of the enemy ship makes an angle ϕ with OP [When the ship is going straight away $\phi = 0$] Prove that in order to allow for the motion of the enemy ship, the axis of the torpedo must be aimed ahead of the ship by an angle δ where

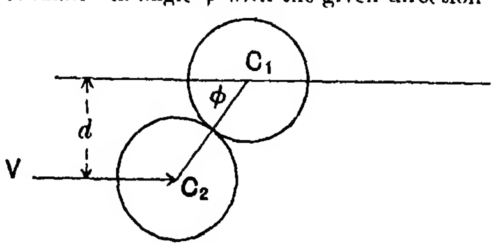
$$\sin\delta = \frac{v}{V}\sin\phi,$$

so that δ is independent of the range OP

Prove that this result is unaffected by a uniform tidal current

Prove further that, at a time t after firing, the locus of the positions relative to the ground of torpedoes fired in all horizontal directions from a fixed point, is a circle whose radius is independent of the tidal current, and find the centre of this circle

vii A smooth perfectly elastic sphere of radius a is at rest on a horizontal table and a second equal sphere is projected with velocity V, *in a given direction, so that the line of centres* at contact makes an angle ϕ with the given direction



The component velocities of the spheres in the given direction are V_1 and V_2 after the collision. Prove that if a large number of such collisions take place for all values of ϕ between $\pm \frac{1}{2}\pi$, and all values of ϕ are regarded as equally probable, then the average value of V_1 or of V_2 will be $\frac{1}{2} V$; but if all values of the displacement d between $\pm 2a$ are regarded as equally probable, then the average value of V_1 is $\frac{2}{3} V$, and of V_2 is $\frac{1}{3} V$.

viii. A running watch is placed on a smooth horizontal surface so that it may be regarded as free to rotate about its own centre of gravity. It is composed essentially of a balance wheel of moment of inertia i, and a body of moment of inertia I connected by a hair spring. The inertia of the hair spring and other parts of the mechanism may be ignored. Determine the angular motion, and shew that in general in addition to its oscillations, the body of the watch rotates steadily with a small uniform spin.

If $I = 110$ gm. (cm.)^2, $i = 0{\cdot}05$ gm. (cm.)^2, shew that the watch gains nearly 20 seconds a day in this position, if its rate is correct when the body is rigidly held.

ix. Define a line of force, and sketch the lines of force of the system of charges e at $(2a, 0, 0)$ and $-\frac{1}{2}e$ at $(\frac{1}{2}a, 0, 0)$.

Prove that the sphere $r = a$ is the equipotential $V = 0$.

Prove that, if the sphere $r = a$ is a conductor at zero potential, the surface density at $(a, 0, 0)$ is $-\dfrac{3e}{4\pi a^2}$, and the mechanical stress on the conductor $\dfrac{9e^2}{8\pi a^4}$ dynes per (cm.)^2.

x. Prove that if no current flows through the galvanometer G of a Wheatstone's bridge (see Figure), then

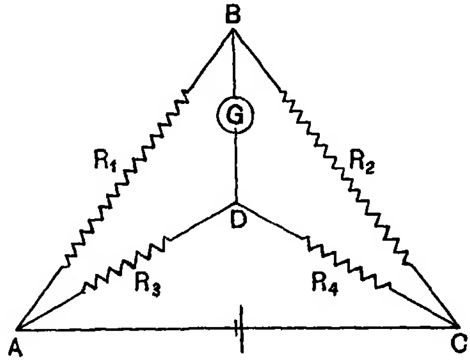
$$\frac{R_1}{R_2} = \frac{R_3}{R_4},$$

and that the battery and galvanometer circuits may be interchanged, without affecting this result.

R_1 and R_2, R_3 and R_4 are pairs of nearly equal resistances. No current flows through G when large resistances X_1 and X_2

are inserted between A, B and B, C in parallel with R_1 and R_2. When R_3 and R_4 are interchanged the necessary large resistances are Y_1 and Y_2. Prove that approximately

$$R_1 = R_2 + \tfrac{1}{2}R_3^2\left(\frac{1}{X_1} + \frac{1}{Y_1} - \frac{1}{X_2} - \frac{1}{Y_2}\right).$$



Shew how this equation may be used to determine R_1, R_2 with great accuracy, when all the resistances are originally known with errors not exceeding 0·1 per cent.

SATURDAY, 5 *June*. 9—12.

1. The base of a haystack is a rectangle of length a and breadth b $(a > b)$, and the vertical sides are of height h. It is thatched with four plane faces sloping at equal angles α to the horizontal. Find the length of the ridge at the top of the stack, and prove that the volume is equal to

$$abh + \tfrac{1}{4}b^2(a - \tfrac{1}{3}b)\tan\alpha.$$

2. Express the radius of curvature of the curve $y = f(x)$ in terms of dy/dx and d^2y/dx^2.

A graph is drawn to shew the relation between space and time in rectilinear motion, one inch representing a feet of space and b seconds of time. Express the velocity and acceleration in terms of the slope of the graph to the time line and the radius of curvature.

If the motion is that of a shot fired vertically upwards at 1600 ft./sec., find a and b so that the initial slope of the graph

may be $\tan^{-1} 2$ and the radius of curvature at the point corresponding to greatest height may be 5 inches.

3 Find the equation of the normal at any point P of the curve $x^3 + y^3 = a^3$.

Prove that the area of the triangle formed by the normal and the axes is equal to the square on the perpendicular from P to one of the bisectors of the angle between the axes.

4. Prove that the middle points of parallel chords of an ellipse are collinear, and find the relation between the eccentric angles of the extremities of conjugate diameters.

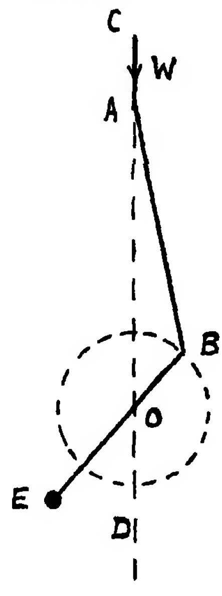
Prove that, if CP' is conjugate to CP and CQ' to CQ, then $P'Q'$ is either parallel to PQ or to the diameter which bisects PQ.

5. Find the equation of the tangent plane at any point of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

Normals are drawn to the ellipsoid making a constant angle α with the plane of xy. Prove that they cut this plane on the ellipse

$$\frac{a^2 + c^2 \tan^2 \alpha}{(a^2 - c^2)^2} x^2 + \frac{b^2 + c^2 \tan^2 \alpha}{(b^2 - c^2)^2} y^2 = 1.$$



6. State the principle of virtual work for a system of forces in equilibrium.

The diagram shews a simple engine mechanism. The point A is pushed downwards by a constant vertical force W, and is free to move smoothly in the vertical line CD passing through O. The crank OB which rotates about O is prolonged to E, where a weight w is attached. Neglecting the weights of the links AB, EB, shew that the positions of equilibrium are given by

$$\theta = 0, \quad \theta = \pi, \quad \theta = \pm\theta_1, \quad \theta = \pi \pm \theta_1,$$

where
$$\sin \theta_1 = \sqrt{\frac{l^2 (aw - Wr)^2 - W^2 r^4}{r^2 aw (aw - 2Wr)}}.$$

Here $AB = l$, $OB = r$, $OE = a$, and θ is the angle AOB.

vii Two smooth uniform spheres of unequal mass impinge directly Shew that there is a net loss of kinetic energy equal to $(1 - e^2)$ times the original kinetic energy of the two spheres relative to their centre of gravity, where e is the coefficient of restitution

A spherical particle is let fall vertically under gravity and after describing a distance h impinges at a point A on a smooth plane inclined at an angle α to the horizontal Shew that the particle ceases to rebound from the plane when it reaches a point B such that

$$AB = 4he \sin \alpha \,(1 - e)$$

viii A train of mass 300 tons is originally at rest on a level track It is acted on by a horizontal force F which uniformly increases with the time, in such a way that $F = 0$ when $t = 0$, $F = 5$ when $t = 15$, F being measured in tons weight, t in seconds When in motion the train may be assumed to be acted on by a frictional force of 3 tons, independent of the speed of the train Find the instant of starting and shew that at $t = 15$, the speed of the train is 0 64 ft per second, whilst the Horse Power required at this instant is about 13

(N B 1 Horse Power = 33000 foot pounds per minute,
$g = 32$ feet per second per second)

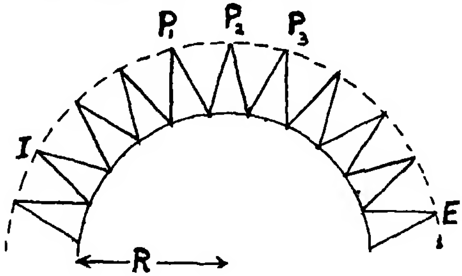
ix A gas enclosed in a containing vessel is allowed to expand from the volume V_1 to the volume V Shew that the work done by the gas on the containing walls is $\int_{V_1}^{V} p\,dV$

A cubic foot of air at 100 lbs per sq in pressure expands down to atmospheric pressure (14 7 lbs per sq in), following Boyle s law Shew that the work done is 27600 ft lbs approximately

x Shew how to find the minimum deviation for a beam of light passing through a prism in a principal plane

The diagram shews a series of equal thin glass prisms of refractive index μ mounted symmetrically round a cylinder of radius R, with their edges parallel to the axis of the cylinder, and lying on the circle $P_1 P_2 P_3$ etc , of radius $(R + h)$ Shew that if the prisms extend completely round the cylinder without gaps between adjacent base edges and if further, $h = R(\mu - 1)$,

then any small object I placed on one of the points $P_1, P_2, P_3, \ldots$ can be seen by an eye E placed near one of the other points P.



SATURDAY, 5 *June*. $1\frac{1}{2}$—$4\frac{1}{2}$.

1. AA' is the straight edge of a horizontal plane $AA'N$, and O, O' are points a distance h below A and A'. If P is any point in space, a distance $H-h$ above $AA'N$, and OP, $O'P$ cut $AA'N$ in points of coordinates x, y, x', y' referred to A, A' as origins and AA' as x-axis, prove that

$$y = y', \quad H = \frac{hB}{x - x'},$$

where $AA' = B$. If the plane is cut in two and the whole system at A' lowered a distance D below A, prove that

$$y' = \frac{Hy}{H+D}, \quad H = \frac{hB + x'D}{x - x'}.$$

2. A sequence of terms $u_0, u_1, u_2, \ldots u_n, \ldots$ is such that any three consecutive terms are connected by the relation

$$6u_{n+1} - 5u_n + u_{n-1} = 0.$$

Given that $u_0 = 1$, $u_1 = 1/6$, find an expression for u_n, and hence deduce that the series $u_0 + u_1 + \ldots + u_n + \ldots$ converges to the sum unity.

3. Shew that the number of permutations of n things taken all at a time, when p of the things are all alike of one kind, q all alike of a second kind, r all alike of a third kind, etc., is

$$n!/p!\,q!\,r! \ldots.$$

N points are taken in space, so that not more than three points lie in any plane, and not more than two in any straight

line. Shew that these points define $N(N-1)/2$ straight lines and $(N-1)!/2$ polygons of N sides.

4. Establish Simpson's Rule that if

$$y = y(x) \equiv a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

then $$\int_0^1 y\,dx = \tfrac{1}{6}\{y(0) + y(1) + 4y(\tfrac{1}{2})\}.$$

Prove that if $y(x)$ also contains a term $a_4 x^4$ the error in still using Simpson's Rule is $\frac{1}{120}a_4$.

5. Assuming that two solutions $y_1(x), y_2(x)$ of the equation $\frac{d^2y}{dx^2} + Qy = 0$ have been found such that $y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} = 1$, verify that a particular integral of

$$\frac{d^2y}{dx^2} + Qy = f(x)$$

is given by the formula

$$y = y_2(x)\int_1^x y_1(\xi) f(\xi)\,d\xi - y_1(x)\int_1^x y_2(\xi).f(\xi)\,d\xi.$$

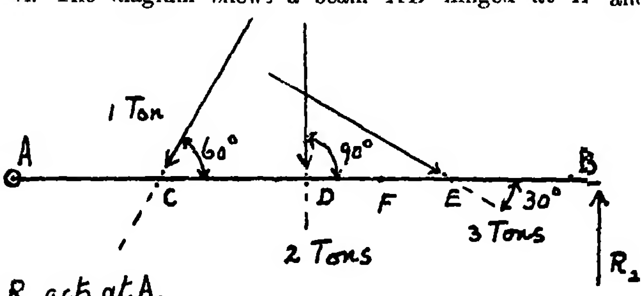
Verify that x^3 and x^{-2} are solutions of the equation

$$\frac{d^2y}{dx^2} - \frac{6}{x^2}y = 0,$$

and hence or otherwise obtain a particular integral of

$$\frac{d^2y}{dx^2} - \frac{6}{x^2}y = x\log x.$$

vi. The diagram shews a beam AB hinged at A and



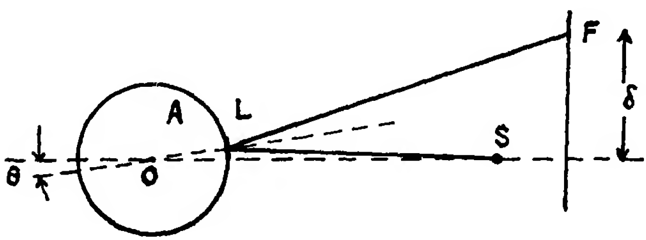
$AC = CD = DE = EB = 5$ feet, $DF = FE$.

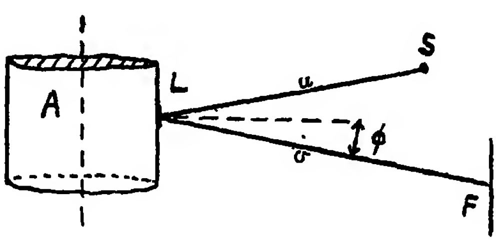
supported at B. Find graphically or otherwise the reactions R_1 and R_2, due to the given system of loads, and the direction of R_1. Shew the Bending Moment diagram, and state the value of the Bending Moment at F.

vii. A particle of mass m moves under the action of a central force km/r^2, where r is the distance of the particle from the centre of force O. Shew that the particle describes a conic, and find the condition that this may be a hyperbola, in terms of the circumstances of projection.

If the effect of the central force is small, so as only to disturb slightly the motion of the particle as it approaches O from an infinite distance away, shew that the total small change α in direction produced by O is given by $\alpha = 2k/V^2D$. Here V is the velocity of the particle at infinity, and D is the perpendicular distance of O from the asymptotes.

viii. The accompanying diagrams shew an arrangement that has been used to measure small and extremely rapid angular





movements of a shaft A. On the shaft is mounted a thin lens L (whose back is silvered) which serves to produce an approximate image of a source of light S at F on a photographic plate.

Shew that corresponding to a small angular movement θ of the shaft, we get a deflection δ on the plate given by

$$\delta = \theta \{2v \cos\phi + r(1 + v/u)\}.$$

Here $OL = r$, $LS = u$, $LF = v$.

ix. Explain the principle of a simple tangent galvanometer.

If $\theta_1, \theta_2, \theta_3$ are the observed deflections when (1) one Grove cell, (2) two such cells in series, (3) two such cells in parallel, are in the galvanometer circuit, prove that

$$2 \tan\theta_1 (\tan\theta_2 + \tan\theta_3) = 3 \tan\theta_2 \tan\theta_3.$$

Shew that if the resistance R of the galvanometer is known, the resistance of a Grove cell can be determined without a knowledge of the galvanometer constant, but that this constant *must* also be known in order to determine the electromotive force of the cell.

x. Explain the function of a guard ring for a cylindrical or parallel-plate condenser.

A condenser is formed of three concentric cylinders of which the inner and outer are connected together. Obtain a formula for the capacity, neglecting end effects, and shew that if the middle plate is 10 cms. long and the radii of the cylinders are 3·9, 4·0, 4·1 cms. the capacity is equal approximately to that of a sphere of 1 metres radius.

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THURSDAY, 2 *June*. 9—12.

A 1. From the circumcentre S of a triangle ABC perpendiculars SD, SE, SF are drawn to the sides BC, CA, AB respectively, and produced to A', B', C' so that D, E, F are the mid-points of SA', SB', SC'; prove that the triangle $A'B'C'$ is equal in all respects to the triangle ABC, and can be derived from it by rotation through two right angles about the common mid-points of AA', BB', CC'.

A 2. Define the inverse of a point and of a curve with respect to a circle S; and prove that, if S' is any circle cutting S orthogonally, S' is its own inverse with respect to S.

Prove that, if a point is inverted any number of times with respect to two circles, all the points so obtained lie on a circle.

A 3. Prove that the centroid of the four points in which a circle meets a parabola lies on the axis of the parabola, and that conversely, if the centroid of four points on a parabola lies on the axis, the four points lie on a circle.

Hence or otherwise prove that, if the axes of two parabolas, which meet in four real points, are perpendicular, the four points lie on a circle, and that the distance of the centre of the circle from the axis of either parabola is half the latus rectum of the other.

A 4. Define the polar of a point with respect to a conic, and prove that if A lies on the polar of B, then B lies on the polar of A.

The polars of a variable point with respect to two given rectangular hyperbolas are at right angles; prove that the locus of the point is in general a circle through the centres of the two hyperbolas, but that, in the special case when the *asymptotes* of one hyperbola are parallel to the *axes* of the other, the locus is a straight line.

A 5. Explain briefly the method of mathematical induction, and give an illustration of its use.

Prove by induction that the sum of n terms of the series

$$x+(x-2)x+(x-4)x(x-1)/2!+(x-6)x(x-1)(x-2)/3!+\dots$$

is
$$x(x-1)\dots(x-n+1)/(n-1)!.$$

A 6. Prove that, if a is an approximation to a root of an equation $f(x)=0$, then $a-f(a)/f'(a)$ is in general a closer approximation.

By applying this formula twice, or otherwise, find to three places of decimals the root near 2 of

$$x^4-12x+7=0.$$

A 7. Give a systematic account of the best way of obtaining the numerical values of the unknown elements in any soluble triangle, including the ambiguous case. Quote the formulae you require, explaining why you prefer them to others.

B 8. Define a differential coefficient, and differentiate *ab initio* $\sin^2 x$ and $\sqrt{x}$.

Differentiate $e^{\sin(\log x)}$.

B 9 A chord is drawn joining the points $x = a$, $a + h$ on e curve $y = f(x)$ If Δ is the difference of the ordinates of e curve and the chord, prove that Δ_m, the numerically eatest value of Δ in the interval, must occur at a point $a + \xi_m$, where ξ_m satisfies the equation

$$f'(a+\xi_m) - \frac{1}{h}\{f(a+h) - f(a)\} = 0$$

Deduce that

$$\Delta_m = \pm \int_0^{\xi_m} \{f'(a+\xi) - f'(a+\xi_m)\}\, d\xi,$$

here the lower limit, 0, may be alternatively h, and hence otain

$\Delta_m < \frac{1}{8}h$ (greatest value of $f''(x)$ in the interval)

It is desired to construct a table of e^x in which interpolation y proportional parts shall be accurate to 0 00005 for $0 < x < 4$ rove that a table of entries for values of x varying by 0 01 ay be inadequate, but that a table of entries varying by 001 is adequate for the purpose

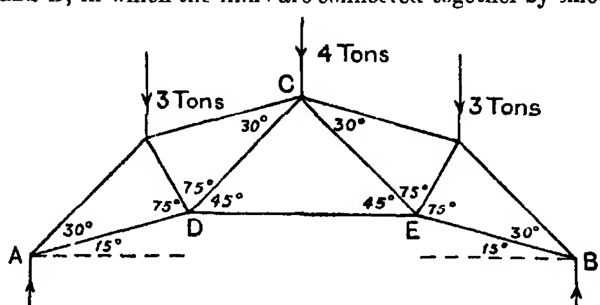
B 10 Sketch the curve

$$y = \frac{a\,x}{a - x},$$

etermining in particular the asymptotes and points of inflexion f any) Evaluate the radius of curvature at the origin

THURSDAY, 2 *June* 1 $\frac{1}{2}$ 4 $\frac{1}{2}$

A 1 The diagram shews a plane framework supported at and B, in which the links are connected together by smooth



hinges. The reactions at A and B are perpendicular to the horizontal line AB. For the given vertical loads, draw the stress diagram, and give the approximate values of the stresses in the different links, stating which are tensional and which compressive stresses.

Check your result for the link DE by taking moments about C for the portion of the framework which lies to the right (or left) of the vertical plane through C at right angles to DE.

A 2. A pair of compasses is used to describe circles on a horizontal piece of paper, and has the legs (of equal length L) always in a vertical plane. The handle at the joint is always vertical. If ϕ be the angle of friction between the compass pencil and the paper, 2α the angle between the legs, and W the vertical pressure on the handle, shew that a horizontal couple $WL \sin\alpha \tan\phi$ must be applied to the handle in drawing a circle, the joint being supposed to be clamped.

When the joint is clamped, the angle 2α at the joint is capable of being increased elastically by a small amount kC when a bending moment C is applied to the joint in the plane of the legs. When a circle is about to be drawn as above, it is found that this increase in 2α takes place as the weight W is applied, provided $\alpha > \phi$. If $\alpha < \phi$, the increase takes place just after the start. Shew that, as a result, the radius of the circle drawn in the latter case is increased by $\frac{kWL^2}{4} \sin 2\alpha$, and explain these phenomena by the laws of friction.

A 3. Explain the terms: *stable*. *unstable*, and *neutral equilibrium*.

A cylinder of radius a whose axis OO' is always horizontal, can roll down a perfectly rough plane inclined at an angle α to the horizontal. The cylinder is eccentrically loaded, so that its centre of gravity G is distant r from OO'. Shew that if $r > a \sin\alpha$, equilibrium is possible for two positions of G relative to OO', and that in each case the angle which the plane $OO'G$ makes with the vertical is $\sin^{-1}\left(\frac{a \sin\alpha}{r}\right)$.

Shew also that only one of these positions gives stable equilibrium, and indicate it.

A 4. Explain the principles of the Conservation of Linear Momentum and Energy.

A shell of mass 1120 lbs., and velocity 1350 f.s., is fired into a railway truck (containing sand) of mass 20 tons, the direction of motion being parallel to the rails. If the shell fails to penetrate the sand, find the velocity given to the truck and account for the conservation of energy in the phenomenon, specifying how much remains kinetic. How far will the truck run against a constant retarding force of 30 lbs. weight per ton?

A 5. A train of weight W lbs. moving at V feet per sec. on the level is pulled with a force of F lbs. against a train resistance of R lbs. Shew that in accelerating from V_0 to V_1 ft. per sec., the distance in feet described by the train is

$$\frac{W}{g}\int_{V_0}^{V_1}\frac{V\,dV}{F-R}.$$

If $W = 300$ *Tons*, $R = 2160 + 1.5\,V^2$, shew that the distance described in slowing down on the level from 45 to 30 miles per hour, with the power shut off, is about 537 feet

($\log_e 10 = 2.303$, $g = 32.2$ ft. per sec. per sec.).

B 6. Prove that the pressure at a point in a fluid is the same in all directions.

Find the surfaces of equal pressure in the liquid partly filling a hollow cylinder of internal radius r, the whole rotating with angular velocity ω as a rigid body about the axis of the cylinder which is vertical.

B 7. State and prove the conditions of equilibrium of a solid floating freely in a liquid.

A hemisphere, radius a, is entirely submerged in liquid of density ρ so that its diametral plane makes an angle θ with the horizontal and has its centre at a depth h. Prove that the resultant force on the curved surface is

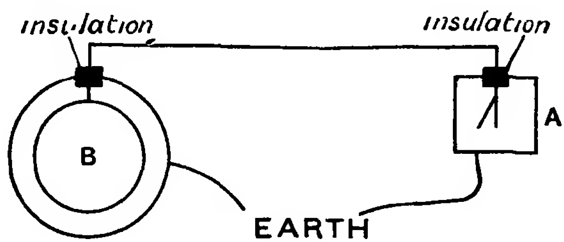
$$\pi a^2\rho g\left\{\tfrac{1}{4}a^2 + h^2 \pm \tfrac{4}{3}ah\cos\theta\right\}^{\frac{1}{2}}.$$

B 8. Prove Gauss' theorem that the integral of normal electric force over any closed surface is equal to 4π times the charge enclosed.

On a certain day the vertical electric force in the atmosphere at the earth's surface was 100 volts per metre and at a height of 1·5 kilometres it was 25 volts per metre. Prove that the total charge in the atmosphere per square kilometre of the earth's surface up to this height was $1 \cdot 99 \times 10^6$ electrostatic units, and that the effect of this charge is to increase the barometric pressure by about 4×10^{-7} dynes per sq. cm., assuming the charge uniformly distributed.

[300 volts = 1 electrostatic unit of potential.]

B 9. The deflection of a gold leaf electroscope A over a certain range is proportional to the charge on it and the change in capacity due to movements of the leaf may be neglected. It is connected to the inner sphere B of a distant spherical air condenser of radii 5 and 6 cms. by a wire which passes through an insulated hole through the outer sphere which is earthed. The deflection of the electroscope is 5 divisions. The outer sphere is then taken in half and removed, and the deflection of the electroscope is 12 divisions. Find the capacity of the electroscope.



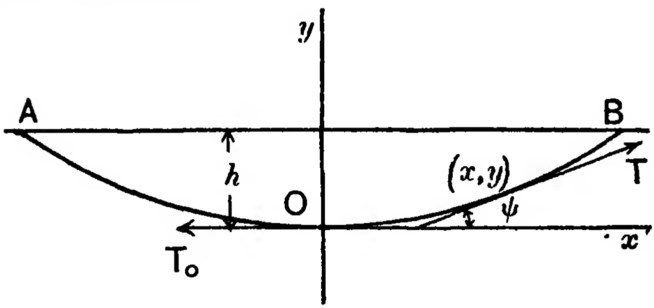
B 10. A galvanometer has a resistance of 1 ohm and each graduation represents 1 milliampere. Find the resistance with which it must be shunted to make it read amperes on the same scale. It is required to use the same instrument as a voltmeter. Find the least resistance which must be connected in series with it, to make its readings accurate within 1 %, if the resistance of the batteries whose voltage it is required to measure never exceeds 2 ohms but is otherwise unknown. On what scale will it then read volts?

FRIDAY, 3 *June*. 9—12.

A 1. A uniform rope AOB (see diagram) rotates steadily with angular velocity ω about AB. If axes are taken in the plane of the rope as shewn, with the origin O at the middle point of the rope, shew that

$$T = \frac{T_0}{\cos\psi}, \quad \frac{1}{R} = \frac{\rho\omega^2}{T_0}(h - y)\cos^2\psi,$$

gravity being neglected.



Here T_0 is the tension at O, T is the tension at the point (x, y) where the tangent makes an angle ψ with Ox, h is the deflection at O, ρ is the mass per unit length, and R is the radius of curvature at (x, y)

Shew also that if the rope is nearly taut, so that $\left(\frac{dy}{dx}\right)^2$ can be neglected, the shape of the rope is given by

$$y = h\left\{1 - \cos\left(x\omega\sqrt{\frac{\rho}{T_0}}\right)\right\}.$$

A 2 AOB is a right angled triangular lamina with the sides OA, OB coinciding respectively with perpendicular axes Ox, Oy The hypotenuse AB is of unit length, and the angle OAB is θ.

Forces act on the sides OA, OB as follows: On OA there are a uniform normal pressure of intensity f_y (in the direction Oy), and a tangential pull of intensity f_t in the direction Ox, both intensities are estimated per unit length of OA. Similarly on OB, there are a uniform normal pressure of intensity f_x (in

the direction Ox), and a tangential pull of intensity f_t in the direction Oy; both intensities are estimated per unit length of OB.

Shew that these forces are equivalent to a single force acting at C the middle point of AB, and that this single force has components P and Q respectively in the direction of BA and perpendicular thereto (away from O), given by

$$P = \frac{(f_x - f_y)}{2} \sin 2\theta + f_t \cos 2\theta,$$

$$Q = f_t \sin 2\theta + f_x \sin^2\theta + f_y \cos^2\theta.$$

B 3. State Newton's laws for the impact of two smooth spheres, explaining the nature of the evidence on which they are based.

Two equal marbles, A and B, lie on a smooth horizontal circular groove at opposite ends of a diameter. A is projected along the groove and at the end of time t impinges on B; shew that the second impact will occur after a further time $2t/e$, where e is the coefficient of restitution.

B 4. An aeroplane has a speed of v m.p.h., and a range of action (out and home) of R miles in calm weather. Prove that in a north wind of w m.p.h. its range of action is

$$\frac{R\,(v^2 - w^2)}{v\,(v^2 - w^2 \sin^2\phi)^{\frac{1}{2}}}$$

in a direction whose true bearing is ϕ. If $R = 200$ miles, $v = 80$ m.p.h. and $w = 30$ m.p.h., find the direction in which its range is a maximum, and the value of the maximum range.

B 5. Prove that, when a particle is moving with velocity v along a plane curve whose radius of curvature at the position of the particle is ρ, the particle experiences an acceleration v^2/ρ along the normal to the curve.

Prove that, in order to allow properly for a curve on a railway line of radius 1320 feet for a train moving at 45 m.p.h., the outer rail must be raised above the inner rail by 5·8 inches. [The rails are 4 ft. $8\frac{1}{2}$ in. apart.]

B 6. A pail containing water is allowed to fall freely from rest under the action of gravity. Prove that the pressure at any point in the water is equal to the atmospheric pressure on the surface of the water.

Describe the effect on a frog swimming (a) horizontally, (b) vertically, in the bucket.

A 7. A particle P of unit mass is acted on by a central force μr directed towards a fixed centre O, where r is the distance OP. Shew that the path of P is in general an ellipse.

Shew also that the acceleration at any instant along the normal to the path is inversely proportional to the velocity v of the particle, and is given by $ab\mu^{\frac{3}{2}}/v$, where a and b are the lengths of the semi-major and semi-minor axes.

A 8. For a system of bodies moving in one plane, define *moment of momentum* about an axis perpendicular to the plane, and shew that if the axis is fixed and the only forces acting on the system from without pass through the axis, the moment of momentum is constant.

A man of mass m stands at A on a horizontal lamina which can rotate freely about a fixed vertical axis O. Originally both man and lamina are at rest. The man proceeds to walk on the lamina, ultimately describes (relative to the lamina) a closed circle having OA $(=a)$ as *diameter*, and returns to the point of starting on the lamina. Shew that the lamina has moved through an angle relative to the ground given by

$$\pi\{1-\sqrt{I/(I+ma^2)}\},$$

where I is the moment of inertia of the lamina about the axis.

B 9. State the law of reflection of light, and establish the form of the image of any solid object formed by a plane mirror.

Three plane mirrors, mutually at right angles, are connected rigidly together. Prove that any ray of light incident on the mirrors will be finally reflected back parallel to its original direction.

B 10. Define the unit points of a single thick lens.

Shew how to use the principal foci and unit points to construct geometrically the image of any given small object near its axis formed by a thick convergent lens.

Prove that if the lens is a complete sphere the unit points coincide at the centre.

FRIDAY, 3 *June*. $1\frac{1}{2}$—$4\frac{1}{2}$.

A 1. Prove that the reciprocal of a circle, with respect to *another circle of centre S, is a conic with S as a focus.*

Two conics have a common focus S and two real common tangents; from a variable point P on one of the common tangents two other tangents PX, PY are drawn to meet the second common tangent in X and Y; prove that XSY is equal to one or other of two constant supplementary angles.

A 2. Prove that, if a straight line is perpendicular at O to each of two intersecting straight lines OA, OB, it is perpendicular to the plane AOB.

A tetrahedron is such that each edge is perpendicular to the edge which it does not meet, prove that the perpendicular from any vertex on the opposite face passes through the orthocentre of that face, and that the four perpendiculars meet in a point.

A 3. Establish a formula for the length of the perpendicular from a point (x', y') to a straight line $ax+by+c=0$, the axes being rectangular, deduce the equations of the bisectors of the angles between two given straight lines.

Find the coordinates of the centre of the circle inscribed in the triangle formed by the straight lines

$$3x+4y-1=0,\ \ 12x+5y-8=0,\ \ 4x-3y+2=0;$$

find also (without solving) the equations determining the centres of the escribed circles.

A 4. Prove that the locus of the foot of the perpendicular from a focus of an ellipse on any tangent is a circle on the major axis as diameter.

Given a focus of a conic, two tangents, and a point on the auxiliary circle, give a geometrical construction for the other focus. Shew also how to construct any number of points on the conic.

A 5. Find the equation of the chord joining two points on the ellipse $x^2/a^2+y^2/b^2=1$, whose eccentric angles are α, β, and deduce the equation of the tangent at a point.

The tangent at one end P of a diameter PP' of an ellipse and any chord $P'Q$ through the other end meet in R, prove that the tangent at Q bisects PR.

A 6. Prove that the number of combinations of n things r together is $n!/\{r!(n-r)!\}$.

A pack of cards is dealt (in the usual way) to four players. One player has just 5 cards of a particular suit; prove that the chance that his partner has the remaining 8 cards of that suit is $1/(4\cdot17\cdot19\cdot37)$.

A 7. If the coefficients of the series

$$u_0 + u_1 x + u_2 x^2 + \dots$$

are connected by the relation

$$u_{n+2} + a u_{n+1} + b u_n = 0,$$

where a and b are independent of n, find u_n and the sum of n terms of the series, u_0 and u_1 being given.

Assuming that the series

$$1 + 8x + 14x^2 + 62x^3 + \dots$$

is of this form, find the coefficient of x^n, and the sum (to ∞) of the series, x being small.

A 8. By means of the expansions of e^x and $\log_e(1+x)$ prove that, when n is large,

$$(1+1/n)^n = e(1 - 1/2n + 11/24n^2 - 7/16n^3 + \dots).$$

Shew that e is given by the formula

$$2e = (1\cdot1)^{10} + (\cdot9)^{-10},$$

with an error of about 0·46 per cent.

B 9. (a) If a be positive, shew that

$$\int_{-1}^{1} \frac{dz}{\sqrt{1 - 2az + a^2}} = 2 \text{ if } a < 1,$$

and has the value $\dfrac{2}{a}$ if $a > 1$.

(b) Shew that if n be greater than unity,

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx,$$

and deduce that

$$\int_0^{\frac{\pi}{2}} \sin^8 x\,dx = \frac{35\pi}{256}.$$

B 10 Shew that the solution of the differential equation

$$\frac{d^2y}{dt^2} + 4y = A \sin pt,$$

which is such that both $y = 0$ and $\frac{dy}{dt} = 0$ when $t = 0$, is

$$y = A\,(\sin pt - \tfrac{1}{2}p \sin 2t)/(4 - p^2)$$

Shew also that is p is made to approach indefinitely near to 2, this solution tends to

$$y = A\,(\sin 2t - 2t \cos 2t)/8$$

SATURDAY, 4 *June* 9–12

A 1 Write down series for $\sin\theta$, $\cos\theta$ in ascending powers of θ

If θ is small expand $\tan\theta$ and $\sec\theta - 1$ in powers of θ as far as θ^4

Is a similar expansion possible for $\cot\theta$?

A 2 Prove that

$$\sum_{p=1}^{n} \sin\frac{2p\pi}{n} = \sum_{p=1}^{n} \cos\frac{2p\pi}{n} = 0,$$

if n is an integer > 1

Extend the result to shew that

$$\sum_{p=1}^{n} \sin\left(\alpha + \frac{2rp\pi}{n}\right) = 0,$$

where r is any positive integer not a multiple of n

A 3 If l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, prove that

$$l_1 = \pm(m_2 n_3 - m_3 n_2), \qquad \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1,$$

explaining the ambiguous signs

Shew that in a given ellipsoid the projection of any diameter on a given axis is proportional to the projection on the corresponding principal plane, of the parallelogram the diagonals of which are any two diameters forming a self-conjugate system with the first

A 4. Shew that the planes $x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)=0$ cut the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad (a>b>c)$$

in real circles, lying on the sphere $x^2+y^2+z^2=b^2$.

Find the condition that the planes cut at right angles.

A 5. Find the angle between two straight lines whose direction cosines are l, m, n, l', m', n'.

The planes OAB, OAC are inclined at an angle of $60°$. The angle $AOB=30°$ and $AOC=45°$. Find to the nearest degree the angle BOC.

B 6. Evaluate:

$$\int\frac{1+\sqrt{a+x}}{1-\sqrt[3]{a+x}}\,dx,\qquad \int\frac{dx}{\sin x},\qquad \int\frac{x^2+x+1}{(x+1)^2(x+2)}\,dx.$$

B 7. A curve $pV^n=C$, where n and C are constants, is limited by the points (p_0, V_0) and (p_1, V_1). Shew that $\int Vdp$ between these limits is numerically n times $\int pdV$ between the same limits. Give the values of these two integrals, examining in particular the case when $n=1$.

B 8. The curve $x=f(y)$ joining two fixed points is rotated round the axis of y. Find an expression for the volume of revolution.

The curve $y=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$ is rotated round Oy. Shew that the volume of revolution is $\sqrt{\pi}/h$.

B 9. (a) Solve $\frac{dy}{dx}-\frac{y}{x}=x^5$.

(b) Express $\frac{d}{dr}\left(r\frac{dv}{dr}\right)$ in terms of $\frac{dv}{dr}$ and $\frac{d^2v}{dr^2}$; hence obtain that solution of the equation

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}=K,$$

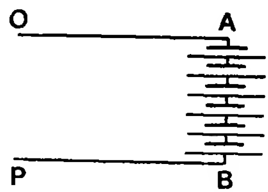
where K is constant, which is zero when $r=a$, and is finite when $r=0$.

A 10. Prove that electrical phenomena inside a closed earthed conductor are not affected by charges outside. Does it make any difference if the conductor is not earthed?

An earthed conductor C encloses two other conductors A and B. A has a given positive charge and B is connected to the positive pole of a battery, the other pole of which is earthed. Sketch the lines of force, distinguishing between the different possible cases.

A 11. State Ohm's Law. Deduce from it an expression for the heat produced by a current.

OA, PB are mains used for charging a battery of cells. The difference of potential between O and P is 25 volts and the resistances of OA, PB are 1 ohm each; 10 cells are being charged in series and the E.M.F. of each (opposing the current) is 2 volts, the internal resistance being 0·2 ohm each. Find the proportion of energy wasted in heat, and shew that it is independent of the resistances.



SATURDAY, 4 *June*. $1\frac{1}{2}$—$4\frac{1}{2}$.

A 1. The top P of a mountain is observed from two points A and B at sea level. If N is the point at sea level vertically below P and $N\hat{A}B = \alpha$, $N\hat{B}A = \beta$, $N\hat{A}P = \theta$ and $N\hat{B}P = \phi$, prove that

$$\cot\phi \sin\beta = \cot\theta \sin\alpha.$$

If the angles α, β, θ are observed and found to be 60° 0′, 45° 0′ and 8° 54′ respectively and AB is 7000 metres, find the height of the mountain. If ϕ were also observed and found to be 7° 24′ how would you use this fact?

A 2. Prove that all the real values of θ between 0 and $\frac{1}{2}\pi$ which satisfy the equation

$$\tan(\cot\theta) = \cot(\tan\theta)$$

are

$$\tfrac{1}{2}\sin^{-1}\frac{4}{(2n+1)\pi}, \quad (n = 1, 2, \ldots),$$

where each inverse sine may be either acute or obtuse. Find to the nearest $1'$ the value of the greatest of these roots less than $\frac{1}{4}\pi$.

A 3. State generally the rule by which the limiting value of $f(x)/\phi(x)$ can be evaluated when x tends to a point a at which $f(a) = 0$ and $\phi(a) = 0$.

Prove that the limiting values, when x tends to 0, for

$$\frac{x - \sin x}{x^3}, \quad \frac{1 - \frac{1}{2}x^2 - \cos x}{x^4},$$

are $\frac{1}{6}$ and $-\frac{1}{24}$ respectively. Verify that these results still hold when $x = iy$, and y is real.

A 4. The parcel post regulations restrict parcels to be such that the length plus girth must not exceed 6 feet, and the length must not exceed $3\frac{1}{2}$ feet.

Determine the parcel of greatest volume that can be sent by post, assuming that the form of the parcel is

(a) a right circular cylinder,

(b) a rectangular box.

How would these results be affected if the greatest length permitted were only $1\frac{3}{4}$ feet?

A 5. Find the equations of the shortest distance between two non-intersecting straight lines.

A straight line is drawn through α, β, γ perpendicular to each of the two straight lines

$$\frac{x-a}{l_1} = \frac{y-\beta}{m_1} = \frac{z-\gamma}{n_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and

$$\frac{x-a}{l_2} = \frac{y-\beta}{m_2} = \frac{z-\gamma}{n_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

$l_1, \ldots, l_2, \ldots$ being actual direction cosines. Shew that the

volume of the tetrahedron formed by α, β, γ and the points where the three lines cut the plane $x=0$ is

$$\frac{\alpha^3 \sin^2\theta}{6 l_1 l_2 (m_1 n_2 - m_2 n_1)},$$

where θ is the angle between the lines (i) and (ii).

A 6. Find the necessary and sufficient conditions that a sphere can be drawn to pass through the circles:

$$x=0,\; y^2+z^2+2g_1y+2h_1z+c_1=0,$$
$$y=0,\; z^2+x^2+2h_2z+2f_2x+c_2=0,$$
$$z=0,\; x^2+y^2+2f_3x+2g_3y+c_2=0.$$

If they are satisfied, find the equation of the projection on the plane $x=0$ of the intersection of the sphere with the plane

$$lx+my+nz=p.$$

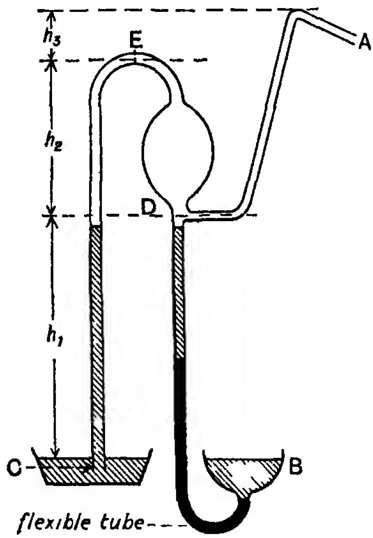
B 7. A mass M is suspended at the lower end of a vertical elastic wire of mass m and length L, suspended at its upper end. The system is caused to execute small vertical oscillations. Assuming that the wire can be treated as uniformly stretched throughout the motion, shew that the kinetic energy of the system is $\frac{\dot{x}^2}{2}(M+\frac{1}{3}m)$, where x is the displacement of the mass M from its equilibrium position. Hence shew that the time of a complete oscillation is $2\pi\sqrt{kL(M+\frac{1}{3}m)}$, where k is such that unit force produces an extension of k units in unit length of the wire.

B 8. The apparatus in the figure contains mercury and air at reduced pressure. It is connected at A to a large reservoir of air at a small pressure equal to that of a height p of mercury, V is the volume of the bulb from D to E and v is the volume per unit height of the tube CE. It is given that $h_1+h_2>P$, where P is the height of the mercury barometer, but $h_2+h_3<P$. The reservoir B is now raised. Prove that the mercury will overflow at E before it flows over into A if

$$p<\frac{v}{V}h_3(h_1+h_2+h_3-P),$$

where $\frac{p^2}{P^2}$ and $\frac{v^2}{V^2}$ are neglected. It is assumed that there is

enough mercury in B to fill the apparatus. The change in level in C can be neglected.



B 9. Four identical convergent thin lenses of focal length f are mounted coaxally in a tube at distances $3f$, $4f$, $4f$ apart. The system is used to form an image of distant objects, with the closest pair of lenses towards the objects. Shew that the system will form an erect real image of distant objects on a ground glass screen, at a distance $13f$ behind the front lens.

If the radius of the lenses and the tube is r and the total length of the tube is $13f$, those distant objects are represented on the screen which lie in a cone of a certain small semi-vertical angle α.

Shew **either** (i) that $\alpha = 3r/7f$, **or** (alternatively), (ii) by tracing the paths of certain rays through the system, that

$$r/3f < \alpha < r/2f.$$

Compare the size of the field of view with that which can be obtained through the same tube without optical fittings, and without moving the eye

B 10. Two thin plano-convex glass lenses

$$(\mu = 1\cdot 5, \quad r = 4 \text{ inches})$$

are set together to form a compound lens with their convex sides in contact. Prove that the focal length of the combination is 4 inches.

If the space between the lenses is filled with water, the focal length is increased to 12 inches. Prove that the refractive index of the water is 4/3.

1922

THURSDAY, 1 *June* 9—12.

A 1. In a triangle ABC, D, E and F are the middle points of the sides BC, CA, AB respectively, and Y, Z are the feet of the perpendiculars from B, C on to the opposite sides. YZ meets FD, DE respectively in M, N.

Prove that the circumcircles of the triangles EYN, FZM have EF as a common tangent and AD as their radical axis.

A 2. *Prove that, if any variable straight line is drawn through* a fixed point P to meet a conic in A and B, and if Q is the harmonic conjugate of P with respect to A and B, the locus of Q is a straight line (the polar of P).

Two conjugate points C, D are taken on the polar of P, and any straight line is drawn through P meeting the conic in A and B. Shew that the points of intersection of the lines AC, BD and of AD, BC lie on the conic and that the straight line joining them passes through P.

A 3. In a quadrilateral $ABCD$, which does not lie in a plane, prove that

(*a*) the middle points of the four sides lie in a plane and determine a parallelogram;

(*b*) the lines joining the middle points of opposite sides and of the diagonals pass through a point and bisect each other there;

(*c*) two opposite sides of any plane quadrilateral $PQRS$, whose vertices lie one on each of the sides of $ABCD$, meet on a diagonal of $ABCD$.

B 4. Prove the identities

$$x^2(y-z)(y^2+z^2-x^2)+y^2(z-x)(z^2+x^2-y^2)$$
$$+z^2(x-y)(x^2+y^2-z^2)=(y-z)(z-x)(x-y)(x+y+z)^2,$$
$$(2y+2z-x)(2z+2x-y)(2x+2y-z)$$
$$+2(x+y+z)(2x^2+2y^2+2z^2-5yz-5zx-5xy)+27xyz=0.$$

B 5. Prove that the arithmetic mean of n positive numbers is equal to or greater than the geometric mean.

Prove that, if a, b, c are three positive numbers, such that each is less than the sum of the other two,

$$(b+c-a)(c+a-b)(a+b-c)<abc.$$

A 6. By De Moivre's theorem (or induction) find an expression for $\tan n\theta$ in terms of $\tan\theta$, n being a positive integer.

By means of the equation $\tan 11x=0$, or otherwise, prove that $\sum_{r=1}^{5}\tan^2\frac{r\pi}{11}=55$,

$$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\tan\frac{5\pi}{11}=\sqrt{11}.$$

B 7. Establish Newton's formulae connecting the sums of powers of the roots of an algebraic equation

$$x^n+p_1x^{n-1}+\dots+p_n=0$$

with the coefficients.

Prove that if $y=\alpha^2(\beta+\gamma)$, where α, β, γ are the roots of

$$x^3+qx+r=0,$$

then
$$y^3-3ry^2+(q^3+3r^2)y-r^3=0.$$

A 8. Shew that the equations of two non intersecting straight lines may be put into the form

$$y=x\tan\alpha,\ z=c \text{ and } y=-x\tan\alpha,\ z=-c$$

A point P moves in the circle $x^2+y^2=r^2$, $z=0$. Shew that the straight lines drawn from P to meet the two given non-intersecting straight lines will meet any given plane $z=h$ in points which lie on a conic.

B 9. Give an account of some useful methods of finding the nth differential coefficient of a function, with an example of each.

Prove (without assuming the series) that the value when $x=0$ of

$$\frac{d^n}{dx^n}\tan^{-1}x$$

is 0, $(n-1)!$, $-(n-1)!$, according as n is of the form $2p$, $4p+1$, $4p+3$, p being an integer.

B 10. Shew how to determine the rectilinear asymptotes of an algebraic curve, without discussing exceptional cases.

Find the asymptotes of the curve given by

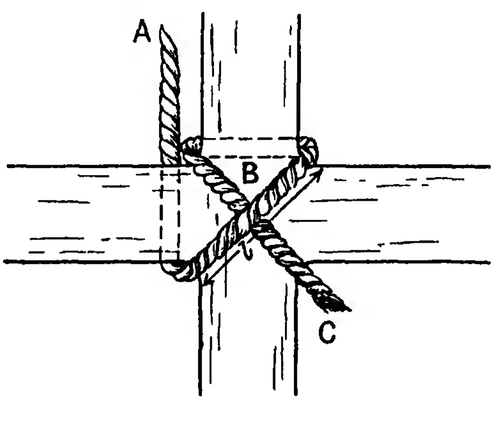
$$x^2y+xy^2+xy+y^2+3x=0.$$

Shew that no part of the curve lies between $x=-4$ and $x=-1$, or between $x=0$ and $x=3$; give a sketch of the curve.

THURSDAY, 1 *June*. $1\frac{1}{2}$—$4\frac{1}{2}$.

A 1. Shew that, if a rope presses against a rough curve and is in limiting equilibrium, the ratio of the tensions at the two ends is $e^{\mu\psi}$, where μ is the coefficient of friction and ψ the total angle turned through by the tangent to the rope.

A rope is passed round a framework of rectangular beams with rounded corners as shewn in the figure. If the diameter of the rope d is small compared with the dimensions of the



framework, shew that the condition that it should not slip when a tension is applied at A, the end C being free, is

$$\mu e^{2\mu\pi} > \frac{l}{8d},$$

where l is measured as in the figure and μ is the coefficient of friction both between the rope and the beams and between the two parts of the rope.

[Assume that the frictions at B act along BC.]

A 2. A stairway is made of n equal uniform smooth rectangular blocks placed on top of each other and each projecting the same distance at the back beyond the one below. The top block is supported from below at its outermost point. Shew that the stairway can stand without mortar if, and only if, $2l > \overline{n-1}\,a$, where $2l$ is the width of each block and a is the width of the tread.

B 3. A sphere collides obliquely with another sphere of equal mass which is initially at rest, both spheres being smooth and perfectly elastic. Shew that their paths after collision are at right angles.

The centres of two such spheres B, C, each of 3 centimetres radius, are at E, F, where $EF = 16$ cms. An equal sphere A is projected with velocity u at right angles to EF, and strikes first B and then C. Its final path is perpendicular to EF. Find the point of contact between A and B. Shew that

$$v_A = 9u/25, \quad v_B = 20u/25, \quad v_C = 12u/25,$$

where v_A, v_B, v_C are the final velocities.

B 4. A force F, which is a function of the abscissa x, acts along the axis of x upon a particle of mass m, which is free to move along that axis. Shew how the change of kinetic energy during any displacement may be represented graphically.

A particle moving along the axis of x has an acceleration Xx towards the origin, where X is a positive function of x which is unchanged when $-x$ is put for x. The periodic time, when the particle vibrates between $x = -a$ and $x = a$, is T. Shew that

$$2\pi/\sqrt{X_1} < T < 2\pi/\sqrt{X_2},$$

where X_1, X_2 are the greatest and least values of X within the range $x = -a$ to $x = a$.

Shew that, when a simple pendulum of length l vibrates through $30°$ on either side of the vertical, T lies between

$$2\pi\sqrt{l/g} \text{ and } 2\pi\sqrt{l/g} \times \sqrt{\pi/3}.$$

B 5. Three particles P, Q, R, each of mass m, attract each other with a force μm^2 (distance). They move on a smooth horizontal plane and, when $t = 0$, are at A, B, C, and are then moving with velocities equal in magnitude and direction to λBC, λCA, λAB. Shew that their centre of gravity, G, remains at rest, and that each particle describes an ellipse about G in the periodic time $T = 2\pi/\sqrt{3\mu m}$.

Shew that the area of each ellipse is $\frac{2}{3}\lambda ST$, where S is the area of ABC.

B 6. State the laws of reflexion and refraction of light.

The quadrilateral $ABCD$ represents, in the two different cases

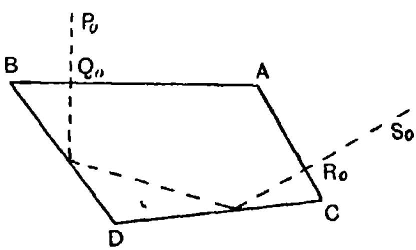


Fig. 1.

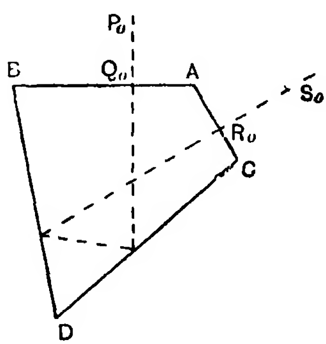


Fig. 2.

shewn in Figs. 1 and 2, the section of a glass prism by a principal plane. The angles at B and C are not necessarily equal. A ray P_0Q_0, in the principal plane, falls normally on AB and, after reflexions at BD and CD, emerges from AC along R_0S_0 at right angles to AC. Find the relation between the internal angles at A and D in each of the two cases.

If, in the case of the first prism, any ray PQ, in the principal plane, fall obliquely upon AB and, after reflexions at BD and CD, emerge from AC along RS, shew that the angle between the *forward* directions of PQ and RS is A, whatever the index of refraction of the glass.

B 7. The distance between two thin converging lenses, each of focal length f, is $2a$, where $a > f$, and O is the centre of the system. Shew that there are two points P_1, P_2 on the axis, such that each coincides with its own image. If their distances from O be x_1, x_2, counted positive when the incident light moves from P to O, shew that

$$\frac{x_1}{a} = \sqrt{\frac{a+f}{a-f}}, \quad \frac{x_2}{a} = -\sqrt{\frac{a+f}{a-f}}.$$

If a small object of height h be placed at P_1, shew that the height of its image is $h(c+a)/(c-a)$, where $c^2 = a^2 - f^2$.

A 8. Give a short account of the experimental evidence for the law of the inverse square in electrostatics.

It is required to hold four equal point charges e in equilibrium at the corners of a square. Find the point charge which will do this if placed at the centre of the square.

A 9. Prove that the mechanical force per unit area on the surface of a conductor is $2\pi\sigma^2$, where σ is the charge per unit area.

A condenser formed of two concentric spheres of radii a and b is divided into two halves by a diametral plane, the inner and outer surfaces being rigidly connected. Prove that the force required to keep the two halves together is

$$\frac{E^2}{8}\left(\frac{1}{b^2} - \frac{1}{a^2}\right),$$

where E and $-E$ are the total charges on the spheres.

A 10. It is required to light 40 electric lamps arranged in parallel. Each lamp needs a potential difference of 100 volts

between its terminals and uses ·16 ampere. If the resistance of the leads to the dynamo is 2 ohms, calculate the E.M.F. required for the dynamo, the energy lost per second in the leads, and that used in the lamps.

FRIDAY, 2 *June*. 9—12.

A 1. Prove that if a vertical force proportional to the mass acts on each particle of a rigid body, the resultant force passes through the same point in the body however the latter is turned.

Find the centre of gravity of a solid cone of vertical angle 2α to the base of which is attached symmetrically a solid hemisphere of radius equal to that of the base, the whole being made of uniform material.

B 2. A particle is projected at time $t=0$ in a fixed vertical plane from a given point S with given velocity $\sqrt{2ga}$, of which the *upward* vertical component is v. Shew that at time $t=2a/v$ the particle is on a fixed parabola (independent of v), that its path touches the parabola, and that its direction of motion is then perpendicular to its direction of projection.

B 3. A light string $ABCDE$, whose middle point is C, passes through smooth rings B, D, which are fixed in a horizontal plane at a distance $2a$ apart. To each of the points A, C, E is attached a mass m. Initially C is held at rest at O, the middle point of BD, and is then set free. Shew that C will come instantaneously to rest when $OC=4a/3$. [The total length of the string is greater than $10a/3$.]

Shew that when C has fallen through $3a/4$ from O, its velocity is $\sqrt{25ag/86}$.

A 4. A tube ABC of mass m is bent at right angles at B. The part AB is horizontal and slides freely through two fixed rings; the part BC is vertical. Particles P, Q, each of mass m, move without friction in AB, BC, and are connected by a string passing over a smooth pulley of negligible mass at B. The system is released from rest. Apply the principles of momentum and energy to shew that, when Q has fallen a distance y from its initial position, its vertical velocity is $\sqrt{6gy/5}$.

Shew that the vertical and horizontal components of the acceleration of Q are $3g/5$ and $g/5$.

A 5. A particle moves under an attraction $\frac{\mu}{r^2}$ per unit mass directed towards a fixed point S. Shew that it describes a conic about S as focus.

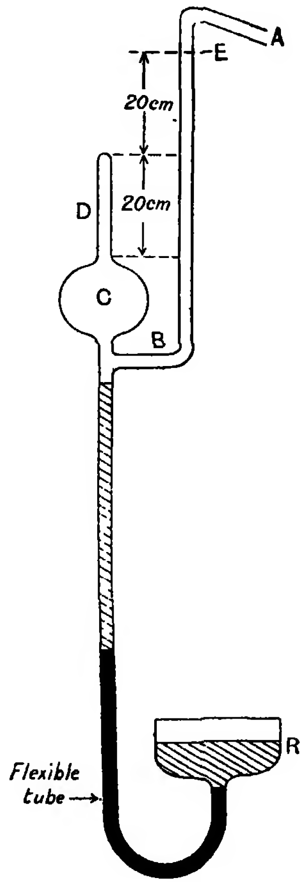
If this conic is a hyperbola, shew that the semi-transverse axis a is given by the equation

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right),$$

where v is the velocity at a point distant r from S, and that the angle between the asymptotes is

$$2\tan^{-1}\frac{pV^2}{\mu},$$

where V is the velocity at infinity and p is the perpendicular from S on an asymptote.



A 6. The apparatus illustrated is used to measure the pressure in a large nearly evacuated vessel connected to it at A.

The reservoir of mercury R is gradually raised until the mercury rises to the fixed mark E, and to a point P in the tube D. The volume of C down to the tube B is 100 c.c. and that of the tube D (length 20 cm.) is ·04 c.c. per cm. The height of E above the top of D is 20 cm. Draw a curve which would enable the pressure in A, reckoned in millimetres of mercury, to be found from the distance of P below the top of D.

B 7. A concave spherical mirror has its centre at O. A ray from a point R, after reflexion

by the mirror at P, meets OR in S. The tangent plane to the mirror at P meets OR in T, and R lies between O and T. If $OR = r$, $SO = s$, $OT = t$, shew that, without any approximation,

$$\frac{1}{r} - \frac{1}{s} = \frac{2}{t}.$$

If the radius $OP = a$, and if $POT = \theta$, find s, when $r = \frac{1}{2}a$. If s is to be not less than $100a$, shew that θ must not exceed $5° 45'$.

B 8. A thin double convex lens, of refractive index μ and radii r, s, separates media of indices μ_1, μ_2, the face of radius r being in contact with the medium μ_1. Obtain a formula connecting the distances of image and object from the lens.

If $\mu > \mu_1 > \mu_2$, shew that there is a point on the axis at a finite distance d from the lens in the medium μ_1 such that it coincides with its own image, and express d in terms of the two focal lengths. If $r = 20$, $s = 50$ cm., and if $\mu = 1{\cdot}5$, $\mu_1 = 1{\cdot}3$, $\mu_2 = 1$, shew that $d = 15$ cm.

A 9. A number of condensers of capacities C_1, C_2 .. C_n are connected (1) in series, (2) in parallel. Shew that, if C is the capacity of the resulting condenser,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots \frac{1}{C_n}$$

in the first case, and $C = C_1 + C_2 + \ldots C_n$ in the second case.

A condenser is formed by a sphere B of radius b inside a concentric sphere A of radius a. A second condenser is formed by a sphere D of radius d inside a concentric sphere C of radius c. Connexions can be made with B, D through small holes in A, C.

Initially A, C are earthed and B, D are at potential U. The sphere C is now insulated and afterwards B is joined to C by a fine wire and D is earthed. The spheres A and C are so far apart that the inductive effect of either on the other may be neglected. Shew that, if V is the final potential of B and C,

$$V = U\frac{ab(c-d) - cd(a-b)}{ab(c-d) + c^2(a-b)}.$$

A 10. Explain the theory of the method of comparing resistances known as the Wheatstone's Bridge.

Two arms AB, AC of such a bridge have resistances 10 ohms each, BD is a standard coil of 5 ohms, and CD a coil of approximately 5 ohms, whose exact resistance is required. The galvanometer (in BC) has a resistance of 50 ohms and can detect 10^{-5} ampere. If the maximum current that may be passed through the resistance BD is $\frac{1}{10}$ ampere, find to two significant figures the smallest difference between the unknown coil and the standard which can be detected.

FRIDAY, 2 *June*. $1\frac{1}{2}$—$4\frac{1}{2}$.

A 1. Find all the solutions of the equations

$$\begin{aligned} x + y + z &= a, \\ x^2 + y^2 + z^2 &= a^2, \\ x^3 + y^3 + z^3 &= a^3. \end{aligned}$$

A 2. By expanding $(2 + x + 1/x)^{2m+n}$, or otherwise, prove that

$$(2m+n)! \sum_{r=0}^{r=m} \frac{2^{2m-2r}}{r!\,(2m-2r)!\,(n+r)!} = \frac{(4m+2n)!}{(2m+2n)!\,(2m)!},$$

m and n being positive integers.

B 3. If α, β are two solutions of the equation

$$a\cos\theta + b\sin\theta = c,$$

not differing by a multiple of four right angles, prove that

$$\tan(\alpha+\beta) = \frac{2ab}{a^2-b^2}.$$

Shew that the result of eliminating θ between the equations

$$\left.\begin{aligned} a\cos\theta + b\sin\theta &= c \\ p\cos\theta + q\sin\theta &= r \end{aligned}\right\}$$

is $$(ap+bq-cr)^2-(a^2+b^2-c^2)(p^2+q^2-r^2).$$

B 4. The bisector of the angle A of a triangle ABC meets BC in D. Shew that

$$AD = 2bc\cos\tfrac{1}{2}A/(b+c).$$

Shew that if the square on AD is equal to three-quarters of the rectangle contained by the sides AC, AB, then the sides of the triangle are in arithmetical progression, and that if the intercept AE on AB by the line through D parallel to AC is one-half BC, then the sides are in harmonical progression.

B 5. Two circles B, B', of radius a, touch one another and have their centres on a diameter of the circle A, of radius $2a$, which they both touch internally. A circle C_1 is drawn to touch A, B and B'; a circle C_2 to touch A, B and C_1, and so on; the circle C_n touching A internally and B, C_{n-1} externally. If the line joining the centre of C_n to the centre of A be of length r_n and make an angle θ_n with the common diameter of B, B', prove that

$$\frac{4a}{r_n} = 3 - \cos\theta_n,$$

$$(2a/r_n - 1)(2a/r_{n+1} - 1) = \sin^2 \tfrac{1}{2}(\theta_n - \theta_{n+1}).$$

Hence prove that $\cot \frac{1}{2}\theta_n = n$, and that the radius R_n of C_n is

$$\frac{2a}{n^2+2}.$$

A 6. Prove that, if the tangent at a point P on a hyperbola meets the asymptotes in T, T', then $TP = PT'$.

Parallels to the asymptotes are drawn through T, T' to meet in Q. Prove that, as P moves on the hyperbola, Q moves on another hyperbola having the same asymptotes.

A 7. Prove that, if the points in which the straight line $l_1x + m_1y + n_1 = 0$ meets the circle $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$, and those in which $l_2x + m_2y + n_2 = 0$ meets

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0,$$

lie on the circle, then

$$2(g_1 - g_2)(m_1n_2 - m_2n_1) + 2(f_1 - f_2)(n_1l_2 - n_2l_1) + (c_1 - c_2)(l_1m_2 - l_2m_1) = 0.$$

A 8. Find the equations of the tangent and normal at the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$.

Prove that the line joining the orthocentres of the two triangles formed respectively by the tangents and by the normals at three points on a parabola is parallel to the axis.

B 9. In a tetrahedron $ABCD$ two opposite edges AC, BD are of length b, and the others are all of length a. Taking B as the origin of rectangular coordinates, BD as the x-axis, and BCD as the xy-plane, find the coordinates of the vertices and the volume of the tetrahedron.

Prove that the edges AC, BD are at right angles, and that the angle between AB and CD is $\cos^{-1}\left(1-\dfrac{b^2}{a^2}\right)$.

B 10. Find the equations to the perpendicular drawn from the point (ξ, η, ζ) to the straight line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

From a point P, whose coordinates are x, y, z, the perpendicular PM is drawn to the straight line through the origin whose direction cosines are l, m, n, and is produced to P', where $PM = MP'$. If the coordinates of P' are x', y', z', shew that

$$\frac{x+x'}{l}=\frac{y+y'}{m}=\frac{z+z'}{n}=2(lx+my+nz).$$

B 11. Find the equations to the normal at a point of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Normals are drawn to the ellipsoid at the points in which it is met by the plane $lx+my+nz=p$. Shew that the points in which they meet the plane $x=0$ lie on a conic.

SATURDAY, 3 *June*. 9—12.

A 1. A steamer is observed from a look-out tower of height h above sea-level. At a given instant the angle of depression is α, after an interval t it is β, and after a further interval t it is γ. Assuming that the steamer is travelling uniformly in a straight line (neglecting the curvature of the earth), prove that its speed is

$$\frac{h}{t}\left\{\tfrac{1}{2}(\cot^2\alpha+\cot^2\gamma)-\cot^2\beta\right\}^{\frac{1}{2}}.$$

Find the least distance of the steamer from the tower.

A 2. Find an expression for the angle of intersection of two spheres which are given by their equations in rectangular Cartesian coordinates.

Prove that the locus of the centre of a sphere which cuts three given spheres at the same angle is a plane.

B 3. Shew that, in general, an approximation, S, to the area A enclosed by a curve $y = f(x)$, the axis of x and the ordinates $x = b$, $x = c$ is given by

$$S = h\left[\tfrac{1}{2}y_0 + y_1 + \ldots + y_{n-1} + \tfrac{1}{2}y_n\right],$$

where $nh = c - b$, and $y_q = f(b + qh)$.

If $\eta = D + E\xi + F\xi^2$, and $\zeta = \eta - \frac{1}{12}h^2 d^2\eta/d\xi^2$, shew that

$$\int_0^h \eta\, d\xi = \tfrac{1}{2}h\,(\zeta_0 + \zeta_1),$$

where ζ_0, ζ_1 are the values of ζ when $\xi = 0$, $\xi = h$.

Hence shew that, in general, a better approximation, T, to the area A is given by

$$T = h\left[\tfrac{1}{2}z_0 + z_1 + \ldots + z_{n-1} + \tfrac{1}{2}z_n\right],$$

where $z = y - \frac{1}{12}h^2 d^2y/dx^2$, and z_q is the value of z when $x = b + qh$.

Find A when $f(x) = x^4$, $b = 0$, $c = 1$, and shew that, when $n = 10$, $S = \cdot 20333$, $T = \cdot 19998$.

B 4. The coordinates of a point Q on the axis of x are ξ, 0, *and those of any other point* P *are* x, y. *Find the value of*

$$V = \int_{-l}^{l} \frac{d\xi}{PQ}.$$

From your result shew that, if $x = \sqrt{b^2 + l^2}\,.\cos\phi$, $y = b\sin\phi$, then

$$V = \log\frac{\sqrt{b^2 + l^2} + l}{\sqrt{b^2 + l^2} - l}.$$

B 5. *If the distance* SP from the focus S to an element ds of the arc of a parabola, whose latus rectum is $4a$, be r, shew that

$$\int ds/r^2 = 4/a.$$

If the normal at P cut the axis in G and the tangent at the vertex in J, shew that

$$\int ds/GJ = \pi\sqrt{2}.$$

In each case the integral is taken over the whole curve.

B 6. The polar equation to the cardioid is $r = a\,(1 + \cos\theta)$. Shew that the circumference of the curve is $8a$.

If A be the area of the curve and V the volume generated when the curve revolves about the line $\theta = 0$, find A and V, and shew that $9V = 16aA$.

B 7. Shew that

$$\int_0^{\pi/2} \frac{\sin x \cos x \, dx}{\cos^2 x + 3\cos x + 2} = \log\frac{9}{8};$$

$$\int_0^1 \frac{dx}{(1-2x^2)\sqrt{1-x^2}} = \frac{1}{2}\log(2+\sqrt{3}).$$

Apply the substitution $y = \tan\frac{1}{2}x$ to obtain the integral

$$\int_0^a \frac{dx}{1 + e\cos x}$$

for the case when $e^2 < 1$, and shew that, when $a = \pi$, the value is

$$\pi/\sqrt{1-e^2}.$$

B 8. Solve the differential equation

$$x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1).$$

If $$x(x-1)\frac{d^2z}{dx^2} - (x-2)\frac{dz}{dx} = x^3(2x-1),$$

and if $z = 0$, and $dz/dx = 0$, when $x = 2$, shew that, when $x = 4$,

$$z = 44 - \log 9.$$

A 9. A sphere of radius a and mass M is loaded so that its centre of gravity G is at a distance c from its centre O, and is suspended by a string attached to a point P of its surface, GP subtending an angle θ at O. The sphere is partly immersed in liquid of density ρ and the tension in the string is $M'g$. Shew that the depth h of O below the surface of the liquid is given by

$$M - M' = \tfrac{1}{3}\pi\rho(a+h)^2(2a-h),$$

and that the inclination of GO to the vertical is

$$\tan^{-1}\frac{M'a\sin\theta}{Mc - M'a\cos\theta}.$$

A 10. Prove that the centre of pressure of a triangle wholly immersed in a liquid with one side parallel to the surface is at the centroid of three masses placed at the middle points of the sides and proportional respectively to their depths below the surface.

One wall of a tank slopes inwards from the bottom at an angle θ to the vertical and contains a triangular trap-door, of weight W, which is hinged about the horizontal side BC, has

the vertex A lower than BC, and can open outwards. The vertical heights of the vertices above the bottom of the tank are a, b, b.

Prove that if water be poured into the tank to a height h so that the trap-door is entirely below the surface, it will remain closed provided that

$$h < \frac{W}{\Delta s} \sin\theta + \tfrac{1}{2}(a+b),$$

Δ being the area of the triangle and s the weight of unit volume of water.

SATURDAY, 3 *June*. $1\frac{1}{2}$—$4\frac{1}{2}$.

A 1. Prove that the equation of a family of coaxal circles can be expressed in the form

$$x^2 + y^2 + 2gx + c = 0,$$

where g varies; distinguish between the cases when c is positive, negative, zero.

Prove that the locus of the poles of a given straight line with respect to the circles of the family is in general a hyperbola, with asymptotes respectively parallel to the radical axis and perpendicular to the given straight line.

A 2. Find the equation of the bisectors of the angles between the straight lines given by

$$Ax^2 + 2Hxy + By^2 = 0.$$

One of the bisectors of the angles between the tangents from a point P to the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

passes through a given point $(d, 0)$ on the major axis. Prove that P lies either on the major axis or on the circle

$$d(x^2 + y^2) - x(a^2 - b^2 + d^2) + d(a^2 - b^2) = 0.$$

A 3. Prove the fundamental property of a pair of conjugate diameters of an ellipse, namely that each diameter bisects all chords parallel to the other.

Prove that the length of the perpendicular from the centre on the chord joining the ends of two conjugate diameters lies between $a/\sqrt{2}$ and $b/\sqrt{2}$, where a and b are the semi-axes.

A 4. Differentiate with respect to x,

$$x/\sqrt{(a^2-x^2)},\ \log\left[\{x+\sqrt{(a^2+x^2)}\}/\{-x+\sqrt{(a^2+x^2)}\}\right],$$
$$(\sin x)^{\tan x},\quad \tan^{-1}(\tanh x).$$

A 5. Assuming that $\tan x$ can be expanded in powers of x, find the expansion up to x^5.

Prove that, if A, A_1, A_2 denote respectively the area of a circle, of an inscribed regular polygon of n sides, and of a similar circumscribed polygon, then

$$n^2(A_2-A_1)\to\pi^2 A,$$
$$n^4(2A_2+A_1-3A)\to\tfrac{2}{5}\pi^4 A,$$

as $n\to\infty$.

A 6. Assuming the formula for total differentiation

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt},$$

find

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2}$$

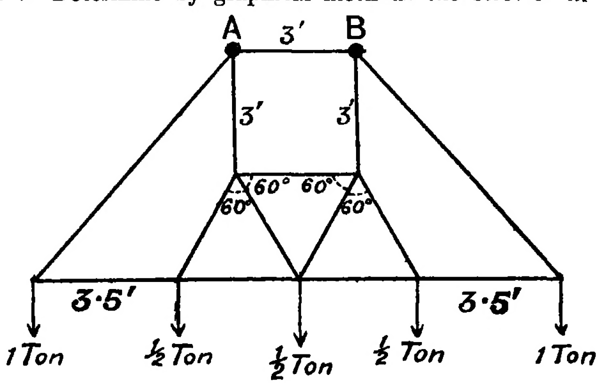
in terms of x, y, when $f(x, y)=0$, and prove that

$$\frac{d^2y}{dx^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-\frac{3}{2}}=\pm\{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2\}(f_x^2+f_y^2)^{-\frac{3}{2}},$$

where suffixes denote partial differentiation.

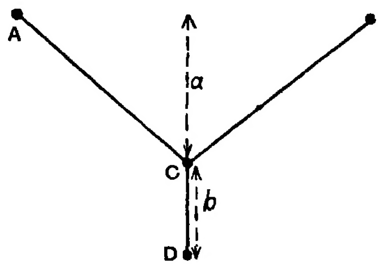
Prove that the radius of curvature at the point $(2a, 0)$, on the curve $\quad (x^2+y^2)^2-2ax(x^2+y^2)-a^2y^2=0$ is $4a/3$.

B 7. Determine by graphical methods the stresses in the



members of the framework shewn, indicating which members are in tension and which are in compression. All the joints are freely hinged, and the whole is supported at A and B.

B 8. A particle is hung at the end of a light string CD knotted at C to two equal light strings AC, CB fastened at points A, B at the same level. Find the equations of motion for small oscillations of the particle in the vertical plane through AB, and in the vertical plane through C perpendicular to AB, and integrate them.



If $a = 3b$, shew that the particle may be made to describe an arc of a parabola.

B 9. A flywheel, turning with average angular velocity p, is acted on by a driving couple $A \sin^2 pt$, and has a constant couple $\frac{1}{2}A$ opposing its motion. Find the least moment of inertia required to make the difference between the greatest and least angular velocities less than $p/100$.

B 10. A uniform rod of mass m and length $2a$ is lying on a smooth horizontal table and is struck a blow B perpendicular to its length at one extremity. Find the velocities with which the two ends of the rod begin to move.

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